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ON MONOTONE RECURSIVE PREFERENCES

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We explore the set of preferences defined over temporal lotteries in an infinite horizon setting. We provide utility representations for all preferences that are both recursive and monotone. Our results indicate that the class of monotone recursive preferences includes Uzawa–Epstein preferences and risk-sensitive preferences, but leaves aside several of the recursive models suggested by Epstein and Zin (1989) and Weil (1990). Our representation result is derived in great generality using Lundberg's (1982, 1985) work on functional equations.

KEYWORDS: Recursive utility, monotonicity, stationarity.

1. INTRODUCTION

INTERTEMPORAL DECISIONS LIE at the heart of many applied economic problems. It is well understood that the analyses of such problems and the related policy recommendations depend critically on the structure of the intertemporal utility functions, and therefore on the underlying decision theoretic assumptions. A popular assumption, first introduced by Koopmans (1960) in a deterministic setting, is stationarity. It implies that an agent can, at all dates, evaluate future prospects using the same history and time-independent preference relation and be time-consistent. In the presence of uncertainty, stationarity is most often complemented by the assumption of recursivity, allowing one to preserve time consistency and history independence. Recursivity is, moreover, extremely useful in applications, as it permits the use of dynamic programming methods. The assumptions of stationarity and recursivity, although of a different nature, are so often coupled together that the single adjective "recursive" is typically used to describe their conjunction.

The so-called *recursive preferences* are the object of analysis in the current paper. More precisely, we study recursive preferences that satisfy another assumption, monotonicity, resulting in the class of *monotone recursive preferences*. As Chew and Epstein (1990, p. 56) explained, monotonicity (called "ordinal dominance" in their paper) roughly "states that

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if two random sequences, C and $\hat{C}$, are such that in every state of the world, the deterministic consumption stream provided in C is weakly preferred to that provided in $\hat{C}$, then C should be weakly preferred to $\hat{C}$." Equivalently, the axiom requires that a decision maker would not choose an action if another available action is preferable in every state of the world. Section 3 of the current paper provides further discussion of the axiom, while Section 6 illustrates its implications in the context of a two-period consumption-savings problem.

Preferences induced by the additively separable expected utility model with exponential discounting, by far the most widely used model of intertemporal choice, are both monotone and recursive. This model has, however, been criticized for its lack of flexibility and in particular for being unable to disentangle risk aversion from the degree of intertemporal substitution. The search for greater flexibility has led researchers to consider either non-recursive or non-monotone preferences. For example, Chew and Epstein (1990, p. 56) explained that "given the inflexibility of the [intertemporal expected utility function] we are forced to choose which of recursivity and ordinal dominance to weaken." Chew and Epstein (1990) explored preferences that are monotone but not necessarily recursive, while Epstein and Zin (1989) considered preferences that are recursive but not necessarily monotone. The latter article has provided a widespread alternative to the standard model of intertemporal choice.

The current paper explores if and how flexibility can be obtained within the set of monotone recursive preferences. The core of the analysis is developed in the risk setting, where preferences are defined over temporal lotteries.[1] Our main finding is that recursive preferences are monotone if and only if they admit a recursive utility representation $U_t = W(c_t, I[U_{t+1}])$ with a time aggregator W and a certainty equivalent I belonging to one of the following two cases:[2]

– $W(c, x) = u(c) + \beta x$ and I is translation-invariant, or
– $W(c, x) = u(c) + b(c)x$ and I is translation- and scale-invariant.

Our results contribute to a growing literature that seeks to understand how alternative models of intertemporal choice differ in their predictions. Notably, the specifications we derive constrain both ordinal preferences and risk preferences.[3] Regarding ordinal preferences, Koopmans (1960), whose analysis was restricted to a deterministic setting, showed that any monotone time aggregator W generates a stationary preference relation. Here, we obtain that only ordinal preferences that can be represented by affine time aggregators can be extended to monotone recursive preferences once risk is introduced. Moreover, the restrictions related to the certainty equivalent I drastically reduce the set of admissible risk preferences. A direct consequence of our results is that the specifications suggested by the general recursive approach of Epstein and Zin (1989) are monotone only in some specific cases, which are detailed in Section 4.3. In particular, the most widely used isoelastic specification of Epstein and Zin (1989) and Weil (1990) is not monotone, unless it reduces to the standard additively separable model of intertemporal choice or the elasticity of intertemporal substitution is assumed to be equal to 1.

[1]Section A of the Appendix extends the analysis to a setting of subjective uncertainty.

[2]From the perspective of period t, the continuation utility U_{t+1} may be random. A certainty equivalent I provides a general way of computing the "expected value" of U_{t+1}. This expected value is then combined with current consumption c_t via the time aggregator W in order to compute U_t. Formal definitions are given in Section 4.1.

[3]We use the terms "ordinal preferences" to refer to preferences over deterministic consumption paths.

As a corollary of our main result, we obtain novel characterizations of two models that have featured prominently in applied work. We do so by restricting attention to preferences à la Kreps and Porteus (1978), that is, to recursive preferences which admit a certainty equivalent of the expected utility form. We find that such preferences are monotone if and only if they admit one of the following two recursive representations:

- $U_t = u(c_t) - \beta\frac{1}{k}\log(E[e^{-kU_{t+1}}])$, or
- $U_t = u(c_t) + b(c_t)E[U_{t+1}]$.

The first case corresponds to the risk-sensitive preferences of Hansen and Sargent (1995), while the second case corresponds to Uzawa–Epstein preferences, which were first introduced by Uzawa (1968) in a deterministic setting and then extended to an uncertain setting by Epstein (1983). Uzawa–Epstein preferences are completely determined by their restriction to deterministic consumption paths. In particular, the degree of risk aversion cannot be modified without affecting ordinal preferences and the elasticity of intertemporal substitution. In contrast, it is known from Chew and Epstein (1991) that the parameter k that enters the recursion defining risk-sensitive preferences leaves ordinal preferences unaffected and has a direct interpretation in terms of risk aversion: the greater the value of k, the greater the risk aversion. We thus reach the conclusion that risk-sensitive preferences are the only Kreps–Porteus recursive preferences that admit a separation of risk and intertemporal attitudes, while being monotone.

The proof of our main result employs techniques that may be of interest outside the scope of this paper. Namely, we show that combining stationarity, recursivity, and monotonicity is only possible if the utility function satisfies a system of *generalized distributivity equations*, that is, equations of the form $f(x, g(y, z)) = h(f(x, y), f(x, z))$. Such equations were studied by Aczél (1966) and solved in great generality by Lundberg (1982, 1985) with methods imported from group theory (see Appendix B.1). Our proof offers an introduction to these methods and shows how they can be applied to the study of recursive utility.

The remainder of the paper is organized as follows. Section 2 introduces our choice setting and Section 3 our axioms. Section 4 presents our main representation result and its corollaries. Section 5 develops some intuition for our main result. In Section 6, we use a consumption-savings example to contrast the consequences of using monotone and non-monotone preferences. Section 7 concludes. The Appendix contains an extension to a setting of subjective uncertainty and the proof of our main result.

2. CHOICE SETTING

Time is discrete and indexed by $t = 0, 1, \ldots$. For the sake of simplicity, we assume that per-period consumption lies in a compact interval $C = [\underline{c}, \overline{c}] \subset \mathbb{R}$ where $0 < \underline{c} < \overline{c}$. The infinite Cartesian product C^∞ represents the space of deterministic consumption streams. To introduce uncertainty, we follow Epstein and Zin (1989) and construct a space of infinite temporal lotteries. We should note that our account of the construction is brief and at times heuristic; the reader is referred to Epstein and Zin (1989) and Chew and Epstein (1991) for the formal details. To proceed, we need a few mathematical preliminaries. The Cartesian product of topological spaces is endowed with the product topology. Given a topological space X, the Borel σ-algebra on X is denoted $\mathcal{B}(X)$. The space of Borel probability measures on X is denoted $M(X)$ and endowed with the topology of weak convergence. As is typical, we identify each $x \in X$ with the Dirac measure on x. When convenient, we can therefore view X as a subset of $M(X)$.

The space D of temporal lotteries is defined in two steps. First, let $D_0 := C^\infty$ and, for all $t \geq 1$, let $D_t := C \times M(D_{t-1})$. For each t, D_t is the set of temporal lotteries for which all uncertainty resolves in or before period t. The second step is to define temporal lotteries for which the uncertainty may resolve only asymptotically. Intuitively, one can visualize such lotteries as potentially infinite probability trees, each branch of which is a consumption stream in C^∞. The formal construction uses the notion of an inverse limit of Parthasarathy (1967). As detailed in Epstein and Zin (1989, pp. 942–943) and Chew and Epstein (1991, p. 355), any element m of D can be defined as a sequence $(m_1, m_2, \dots) \in \prod_{t=1}^\infty D_t$ where the elements m_t differ only in the timing of resolution of uncertainty.[4] Such a sequence delivers increasingly finer approximations for the temporal lottery m. As a subset of $\prod_{t=1}^\infty D_t$, the set D inherits the relative product topology.

It is known from Epstein and Zin (1989) that the set D is homeomorphic to $C \times M(D)$. Subsequently, we write (c, m) for a generic temporal lottery in D. There is a clear intuition for this homeomorphism: Each temporal lottery can be decomposed into a pair (c, m) where $c \in C$ represents initial consumption, which is certain, and $m \in M(D)$ represents uncertainty about the future, that is, about the temporal lottery to be faced next period. Since we identify D with a subset of $M(D)$, we can also write $(c_0, (c_1, m)) \in D$ for a temporal lottery that consists of two periods of deterministic consumption, c_0 and c_1, followed by the lottery $m \in M(D)$. More generally, for any consumption vector $c^t = (c_0, \dots, c_{t-1}) \in C^t$ and $m \in M(D)$, the temporal lottery $(c_0, (c_1, (c_2, (\dots, (c_{t-1}, m)))) \dots) \in D$ is one that consists of t periods of deterministic consumption followed by the lottery m. For simplicity, we shorten the last expression by writing $(c^t, m) \in D$ for such a lottery.

Being a space of probability measures, $M(D)$ is a mixture space. We write $\pi m \oplus (1 - \pi)m' \in M(D)$ for the mixture of $m, m' \in M(D)$ given $\pi \in [0, 1]$.[5] The mixture of n lotteries $(m_i)_{1\le i\le n}$ with a probability vector $(\pi_i)_{1\le i\le n}$ will be denoted $\bigoplus_{i=1}^n \pi_i m_i$.

3. AXIOMS

The behavioral primitive in this paper is a binary relation $\succeq$ on the space D of temporal lotteries. In this section, we introduce the main axioms we impose on this relation. The first two are standard.

AXIOM 1—Weak Order: *The binary relation $\succeq$ is complete and transitive.*

AXIOM 2—Continuity: *For all $(c, m) \in D$, the sets $\{(\hat{c}, \hat{m}) \in D | (\hat{c}, \hat{m}) \succeq (c, m)\}$ and $\{(\hat{c}, \hat{m}) \in D | (c, m) \succeq (\hat{c}, \hat{m})\}$ are closed in D.*

The next axiom, Recursivity, which ensures that ex ante choices remain optimal when they are evaluated ex post, is taken from Chew and Epstein (1990).

AXIOM 3—Recursivity: *For all $n, t > 0$, consumption vectors $c^t \in C^t$, temporal lotteries $(c_i, m_i), (\hat{c}_i, \hat{m}_i) \in D$, $i = 1, 2, \dots, n$, and $(\pi_1, \dots, \pi_n) \in (0, 1)^n$ such that $\sum_i \pi_i = 1$, if, for every $i = 1, \dots, n$,*

$$\big(c^t, (c_i, m_i)\big) \succeq \big(c^t, (\hat{c}_i, \hat{m}_i)\big), \tag{1}$$

[4]More precisely, for any t, the temporal lottery $m_t \in D_t$ has to be obtained from the lottery $m_{t+1} \in D_{t+1}$, by translating the resolution of uncertainty that takes place in period $t + 1$ to period t.

[5]In particular, $\pi m \oplus (1 - \pi)m'$ is the probability measure in $M(D)$ such that $[\pi m \oplus (1 - \pi)m'](B) = \pi m(B) + (1 - \pi)m'(B)$ for every Borel subset B of D.

then

$$\left(c^t, \bigoplus_{i=1}^{n} \pi_i(c_i, m_i)\right) \succeq \left(c^t, \bigoplus_{i=1}^{n} \pi_i(\hat{c}_i, \hat{m}_i)\right). \tag{2}$$

The latter ranking is strict if, in addition, one of the former rankings is strict.

The next two axioms, History Independence and Stationarity, are complementary assumptions expressing Koopmans' (1960, p. 294) idea that "the passage of time does not have an effect on preferences."

AXIOM 4—History Independence: *For all* $c, \hat{c} \in C$ *and* $m, \hat{m} \in M(D)$, $(c, m) \succeq (c, \hat{m})$ *if and only if* $(\hat{c}, m) \succeq (\hat{c}, \hat{m})$.

AXIOM 5—Stationarity: *For all* $c_0 \in C$ *and* $(c, m), (\hat{c}, \hat{m}) \in D$,

$$\big(c_0, (c, m)\big) \succeq \big(c_0, (\hat{c}, \hat{m})\big) \quad \textit{if and only if} \quad (c, m) \succeq (\hat{c}, \hat{m}).$$

Following Chew and Epstein (1991), we refer to preferences satisfying Axioms 1 through 5 as *recursive preferences*.[6]

The next axiom requires that, in the absence of uncertainty, higher consumption is always better. To state it, let $\geq$ denote the usual pointwise order on C^∞.

AXIOM 6—Monotonicity for Deterministic Prospects: *For all* $c^\infty, \hat{c}^\infty \in C^\infty$, *if* $c^\infty \geq \hat{c}^\infty$, *then* $c^\infty \succeq \hat{c}^\infty$. *The latter ranking is strict whenever* $c^\infty \gneq \hat{c}^\infty$.

The next and final axiom is central to the analysis of this paper.

AXIOM 7—Monotonicity: *For all* $n, t > 0$, *consumption vectors* $c^t, \hat{c}^t \in C^t$, *consumption streams* $c_i^\infty, \hat{c}_i^\infty \in C^\infty$, $i = 1, 2, \ldots, n$, *and* $(\pi_1, \ldots, \pi_n) \in [0, 1]^n$ *such that* $\sum_i \pi_i = 1$, *if, for every* $i = 1, \ldots, n$,

$$\big(c^t, c_i^\infty\big) \succeq \big(\hat{c}^t, \hat{c}_i^\infty\big), \tag{3}$$

then

$$\left(c^t, \bigoplus_{i=1}^{n} \pi_i c_i^\infty\right) \succeq \left(\hat{c}^t, \bigoplus_{i=1}^{n} \pi_i \hat{c}_i^\infty\right). \tag{4}$$

Monotonicity corresponds to the notion of Ordinal Dominance in Chew and Epstein (1990). It is noteworthy that if the consumption levels during the first t periods are identical, that is, if $c^t = \hat{c}^t$, then the requirement in equation (4) is implied by Recursivity. Monotonicity extends the requirement to the case when $c^t \neq \hat{c}^t$. Another important observation about Monotonicity is that, as in the statement of Recursivity, the consumption

[6]Unlike us, Chew and Epstein (1991) adopted $M(D)$ as the domain of choice for their work on recursive preferences. The difference is immaterial since any recursive preference relation on D extends uniquely to a recursive preference relation on $M(D)$. We should note, however, that if we had chosen $M(D)$ as the domain of choice, then History Independence and Stationarity could have been combined into a single assumption stating that for all $m, \hat{m} \in M(D)$ and all $c \in C$, $m \succeq \hat{m}$ if and only if $(c, m) \succeq (c, \hat{m})$. It is for this reason that we often use the term "stationarity" to mean the conjunction of Axioms 4 and 5.

streams (c^t, c_i^∞) and $(\hat{c}^t, \hat{c}_i^\infty)$ are mixed at the *same* date t on both sides of equation (4). This explains why this notion of monotonicity allows for non-trivial attitudes toward the timing of resolution of uncertainty, which is necessary to achieve a separation of risk and intertemporal attitudes. See Section 4.4 for a detailed discussion of this point.

Monotonicity is a consistency requirement between preferences over temporal lotteries and preferences over deterministic consumption streams. Axiom 7 stipulates that a temporal lottery be preferred whenever it provides a better consumption stream in every state of the world. Monotonicity is satisfied by the standard additively separable model of intertemporal choice and more generally by the recursive preferences axiomatized in Epstein (1983), as a direct consequence of the von Neumann–Morgenstern independence axiom. In a setting of subjective uncertainty, the axiom is found in Epstein and Schneider (2003b), Maccheroni, Marinacci, and Rustichini (2006), and Kochov (2015). It is noteworthy that, in those papers, a stronger version of the axiom is actually used. See Axiom A.7 in Appendix A and the discussion therein. By comparison, the models introduced in the seminal papers of Kreps and Porteus (1978) and Selden (1978) are typically non-monotone, an aspect which is not discussed in these papers.

We should stress that Monotonicity, as well as Recursivity, implies indifference to some forms of uncertainty, reflecting the underlying separability properties implied by these assumptions. Consider, for example, the case of two different consumption streams (c, c_1^∞) and (c, c_2^∞), such that $(c, c_1^\infty) \sim (c, c_2^\infty)$. Note that these consumption streams start with the same first-period consumption c. If Monotonicity or Recursivity (or both) holds, we have $(c, \pi c_1^\infty \oplus (1-\pi)c_2^\infty) \sim (c, c_1^\infty)$ for every $\pi \in (0, 1)$. Thus, there is indifference between a risky temporal lottery and a degenerate lottery, whatever the agent's degree of risk aversion. This may, of course, be seen as disputable: one may argue that an agent who strongly dislikes risk should strictly prefer the deterministic consumption stream (c, c_1^∞) to the temporal lottery $(c, \pi c_1^\infty \oplus (1-\pi)c_2^\infty)$, which allows for future per-period consumption levels to be uncertain. Aversion to such risk is, however, ruled out whenever Monotonicity or Recursivity is assumed. A similar feature also appears when Monotonicity is used in a static setting with a set of outcomes that is not totally ordered (meaning that indifference between different outcomes is possible). For example, consider applying the expected utility theory of von Neumann and Morgenstern to a setup where outcomes are multidimensional consumption bundles. Then, the degree of concavity of the utility index does not reflect aversion to inequalities in the actual outcomes, but aversion to inequalities in the welfare levels associated with those outcomes. More generally, Monotonicity implies that substituting a possible outcome of a lottery with another outcome which is considered equally good leaves the evaluation of the lottery unaffected. But, as soon as the set of outcomes is not totally ordered, this requirement embeds a non-trivial separability property.[7] Our axiom makes no exception in this respect.

As with any separability assumption, one may wonder whether Monotonicity is appealing or excessively restrictive. Our aim is not to take a position on this point but to explore the flexibility that remains when Monotonicity is introduced. The current paper contributes to the literature by fully characterizing the class of monotone recursive preferences. We should also mention that there has been little discussion of the implications of Monotonicity within the type of specific intertemporal decision problems that arise in applications. This is in spite of the fact, which is made clear by our results, that

[7]This is also the case when Monotonicity is formulated in a setting of subjective uncertainty, à la Savage, with a set of consequences that is not totally ordered. A particular example is the setting of Anscombe and Aumann (1963), in which the set of consequences is that of roulette lotteries.

Monotonicity is a key difference between some of the main utility specifications used in practice. For example, the problem of saving under uncertainty has been addressed with monotone specifications in Drèze and Modigliani (1972) and Kimball (1990), and with non-monotone preferences in Kimball and Weil (2009), but there is no discussion on the potential impact of monotonicity breakdowns. Section 6 provides insights into the role of Monotonicity in a standard consumption-savings problem. The discussion illustrates that the restrictions imposed by Monotonicity come with the advantage of providing unambiguous and intuitive conclusions about the role of risk aversion.

The formal statements of Recursivity and Monotonicity exhibit some similarities. Actually, both axioms could be combined into a single axiom, looking like Axiom 3, except that the initial t periods of consumption would not be required to be the same on the left- and right-hand sides of equations (1) and (2). To be precise, we would need to state that if $(c^t, (c_i, m_i)) \succeq (\hat{c}^t, (\hat{c}_i, \hat{m}_i))$ for all i, then one should have $(c^t, \bigoplus_{i=1}^n \pi_i(c_i, m_i)) \succeq (\hat{c}^t, \bigoplus_{i=1}^n \pi_i(\hat{c}_i, \hat{m}_i))$. Recursivity only assumes that the implication has to hold when the consumption levels during the first t periods are identical, that is, when $c^t = \hat{c}^t$. Monotonicity relaxes this restriction, but constrains the (c_i, m_i) and $(\hat{c}_i, \hat{m}_i)$ to be deterministic consumption paths, so that the ranking in (3) relates to ordinal preferences only. Although merging Recursivity and Monotonicity into a single axiom would condense the presentation of our results, we preferred to state them separately, as they refer to conceptually different behavioral restrictions. Recursivity, on the one hand, guarantees that the preference relation leads to time-consistent behavior. The restriction that $c^t = \hat{c}^t$ is natural in that perspective, since for an agent at time t, the past is reflected in c^t and a choice between two different past consumption streams is obviously not available. Monotonicity, on the other hand, is a consistency requirement between ordinal and risk preferences. Monotonicity has to be thought of as constraining the behavior of an agent in period 0, when she has to choose between different possible temporal lotteries. The lotteries to be compared can provide different consumption levels during the first t periods and there is no reason to introduce the restriction $c^t = \hat{c}^t$ in the formulation of Monotonicity.

Finally, note that, as formulated in Axiom 7, Monotonicity is restricted to temporal lotteries that resolve in a single period of time. Bommier and Le Grand (2014) showed that a stronger notion of monotonicity can be formulated, extending the consistency requirement to lotteries that resolve sequentially over many periods. This stronger notion builds on the work of Segal (1990), who provided such an extension for lotteries that resolve in two periods of time. We decided not to pursue this direction here since the extension is quite involved and since our main results can be obtained using only the weaker axiom. We should stress, however, that every preference relation that satisfies Axioms 1 through 7 is monotone in the stronger sense of Bommier and Le Grand (2014).

In the remainder of the paper, we refer to preferences satisfying Axioms 1 through 7 as *monotone recursive preferences*.

4. REPRESENTATION RESULTS

4.1. *Monotone Recursive Preferences*

Our main representation result uses the notion of a certainty equivalent. Formally, a *certainty equivalent* I is a mapping from $M(\mathbb{R}_+)$ into $\mathbb{R}_+$ which is continuous, increasing with respect to first-order stochastic dominance, and such that $I(x) = x$ for every $x \in \mathbb{R}_+$.[8] Informally, one can think of I as specifying an "expected value" to each probability distribution over the reals.

[8] Recall that we abuse notation and identify a degenerate probability distribution on x with x itself.

Two additional properties of certainty equivalents play a major role in the subsequent analysis. For every $x \in \mathbb{R}_+$ and $\mu \in M(\mathbb{R}_+)$, let $\mu + x$ be the probability measure in $M(\mathbb{R}_+)$ such that $[\mu + x](B + x) = \mu(B)$ for every set $B \in \mathcal{B}(\mathbb{R}_+)$. Similarly, for every $\lambda \in \mathbb{R}_+$ and $\mu \in M(\mathbb{R}_+)$, let $\lambda\mu$ be the probability measure such that $[\lambda\mu](\lambda B) = \mu(B)$ for every $B \in \mathcal{B}(\mathbb{R}_+)$.[9] In words, $\mu + x$ is obtained from μ by adding x to each y in μ's support, while $\lambda\mu$ is obtained from μ by scaling each y in μ's support by λ. A certainty equivalent I is *translation-invariant* if $I(x + \mu) = x + I(\mu)$ for all $x \in \mathbb{R}_+$ and $\mu \in M(\mathbb{R}_+)$. It is *scale-invariant* if $I(\lambda\mu) = \lambda I(\mu)$ for all $\lambda \in \mathbb{R}_+$ and $\mu \in M(\mathbb{R}_+)$. Translation invariance has an obvious analogue in the notion of constant *absolute* risk aversion, while scale invariance is related to the notion of constant *relative* risk aversion. In what follows, however, certainty equivalents are applied to distributions of utility levels rather than consumption levels. Thus, the two invariance properties have no direct implications in terms of risk attitudes with respect to consumption.

We proceed by recalling an important result from Chew and Epstein (1991). It delivers a representation for the class of all recursive, but non-necessarily monotone, preferences. Given a function $U : D \to [0, 1]$ and a probability measure $m \in M(D)$, define the *image measure* $m \circ U^{-1} \in M([0, 1])$ by letting

$$[m \circ U^{-1}](B) := m(\{(\hat{c}, \hat{m}) \in D | U(\hat{c}, \hat{m}) \in B\}), \quad \forall B \in \mathcal{B}([0, 1]). \tag{5}$$

It is known from Theorem 3.1 of Chew and Epstein (1991) that the following holds:

LEMMA 1—Recursive Preferences: *A binary relation $\succeq$ on D satisfies Axioms 1 through 5 if and only if it can be represented by a continuous utility function $U : D \to [0, 1]$ such that, for all $(c, m) \in D$,*

$$U(c, m) = W(c, I(m \circ U^{-1})), \tag{6}$$

where $I : M(\mathbb{R}_+) \to \mathbb{R}_+$ is a certainty equivalent and $W : C \times [0, 1] \to [0, 1]$ is a continuous function, strictly increasing in its second argument.

We call the representation in (6), which we denote as (U, W, I), a *recursive representation for $\succeq$*. The function W is called a *time aggregator* and I a *certainty equivalent*. Faced with a temporal lottery (c, m), the individual first evaluates the uncertain future by assigning the value $I(m \circ U^{-1})$ to the distribution $m \circ U^{-1}$ of continuation utilities; this value is then combined with current consumption c via W, so as to compute the overall utility of the lottery (c, m).

We are ready to state the main result of our paper.

PROPOSITION 1—Monotone Recursive Preferences: *A binary relation $\succeq$ on D fulfills Axioms 1 to 7 if and only if it admits a recursive representation (U, W, I) such that either:*

1. $W(c, x) = u(c) + \beta x$ *and I is translation-invariant, where $\beta \in (0, 1)$ and $u : C \to [0, 1]$ is a continuous, strictly increasing function such that $u(\underline{c}) = 0$ and $u(\overline{c}) = 1 - \beta$, or*
2. $W(c, x) = u(c) + b(c)x$ *and I is translation- and scale-invariant, where $u, b : C \to [0, 1]$ are continuous functions such that $b(C) \subset (0, 1)$, the functions u and $u + b$ are strictly increasing, and $u(\underline{c}) = 0$, $u(\overline{c}) = 1 - b(\overline{c})$.*

[9]As is standard, $B + x$ denotes the set $\{y + x : y \in B\}$, and λB denotes the set $\{\lambda y : y \in B\}$.

The formal proof of Proposition 1 is given in Appendix B. Some intuition is provided in Section 5. One of the main lessons of our result is that Monotonicity greatly restricts the admissible specifications of intertemporal utility functions and is thus a powerful criterion to consider. Indeed, the work of Chew and Epstein (1991) facilitated the construction of recursive intertemporal utility functions by allowing one to select a certainty equivalent I from the large literature on *atemporal* non-expected utility preferences and integrating it into an intertemporal utility function via the recursion in (6). However, the question arose as to which specifications of the certainty equivalent I and of the time aggregator W should be used, and what their implications for *intertemporal* behavior would be. The literature on this matter is active and growing. In their comments to Backus, Routledge, and Zin (2005), a paper that surveys the literature on recursive utility, both Hansen and Werning focused on this problem, with Werning in particular emphasizing the need for more work aimed at discriminating among alternative utility specifications. Proposition 1 offers a partial answer to this question.

4.2. *Kreps–Porteus Recursive Preferences*

This section focuses on the case when the recursivity axiom is replaced by the stronger independence axiom of Kreps and Porteus (1978) provided below.

AXIOM 3*—Independence: *For all* $n, t > 0$, *consumption vectors* $c^t \in C^t$, *temporal lotteries* $m_i, \hat{m}_i \in D$, $i = 1, 2, \dots, n$, *and* $(\pi_1, \dots, \pi_n) \in (0,1)^n$ *such that* $\sum_i \pi_i = 1$, *if, for every* $i = 1, \dots, n$,

$$(c^t, m_i) \succeq (c^t, \hat{m}_i), \tag{7}$$

then

$$\left(c^t, \bigoplus_{i=1}^{n} \pi_i m_i\right) \succeq \left(c^t, \bigoplus_{i=1}^{n} \pi_i \hat{m}_i\right). \tag{8}$$

The latter ranking is strict if, in addition, one of the former rankings is strict.

Independence is stronger than Recursivity, since Axiom 3 is obtained from Axiom 3* by restricting the lotteries $m_i, \hat{m}_i \in M(D)$ in (7) and (8) to be degenerate, that is, to be elements of D. Axiom 3*, together with Axioms 1, 2, 4, and 5, implies that every recursive representation (U, W, I) of $\succeq$ is such that the certainty equivalent is of the expected utility kind: $I = \phi^{-1} E \phi$, where E is the standard expectation operator and ϕ an increasing and continuous function.[10] Combined with Proposition 1, we obtain the following result.

PROPOSITION 2—Monotone Kreps–Porteus Recursive Preferences: *A preference relation* $\succeq$ *fulfills Axioms 1, 2, 3*, 4, 5, 6, and 7 if and only if it admits a representation* (U, W, I) *such that either*:

1. Risk-sensitive case: $W(c, x) = u(c) + \beta x$ *and* $I = \phi^{-1} E \phi$ *where* $\phi(x) = \frac{1-\exp(-kx)}{k}$, $k \in \mathbb{R} \setminus \{0\}$ *or* $\phi(x) = x$, $\beta \in (0,1)$, *and* $u : C \to [0,1]$ *is a continuous, strictly increasing function such that* $u(\underline{c}) = 0$ *and* $u(\overline{c}) = 1 - \beta$, *or*

[10] For any $\mu \in M(\mathbb{R}_+)$, we define $E[\mu] = \int_{\mathbb{R}^+} x \mu(dx)$. More generally, for any strictly increasing function ϕ, we define $\phi^{-1} E \phi$ as: for any $\mu \in M(\mathbb{R}_+)$, $(\phi^{-1} E \phi)[\mu] = \phi^{-1}(\int_{\mathbb{R}^+} \phi(x) \mu(dx))$.

2. Uzawa–Epstein case: $W(c, x) = u(c) + b(c)x$ *and* $I = E$ *where* $u, b : C \to [0, 1]$ *are continuous functions such that* $b(C) \subset (0, 1)$, *the functions* u *and* $u + b$ *are strictly increasing, and* $u(\underline{c}) = 0$, $u(\overline{c}) = 1 - b(\overline{c})$.

PROOF: We know that every recursive representation (U, W, I) of $\succeq$ is such that $I = \phi^{-1}E\phi$, for some increasing and continuous function ϕ. Proposition 1 implies two cases. In the first one, the certainty equivalent I is translation-invariant, implying that ϕ is of the constant absolute risk aversion kind (i.e., ϕ exponential or linear). In the second case, I must be scale- and translation-invariant. The function ϕ must then be linear, which implies the Uzawa–Epstein case. *Q.E.D.*

Proposition 2 shows that Uzawa–Epstein preferences and the risk-sensitive preferences of Hansen and Sargent (1995) are the only Kreps–Porteus recursive preferences that are monotone. It is important to observe that among the two cases obtained in Proposition 2, only the first affords a separation of risk and intertemporal attitudes. In the risk-sensitive case, one can vary the parameter k so as to change risk preferences, without affecting preferences over deterministic prospects and in particular the elasticity of intertemporal substitution. By comparison, Uzawa–Epstein preferences are fully determined by their restriction to C^∞ and are therefore not flexible enough to explore the role of risk aversion.

4.3. *Epstein–Zin–Weil Preferences*

A very popular class of preferences that affords a separation of risk and intertemporal attitudes was introduced in Epstein and Zin (1989). While this seminal paper adopts the general recursive representation shown in Lemma 1 as a starting point, Epstein and Zin (1989) focused their analysis on recursive preferences with a representation $(U^{\text{EZ}}, W^{\text{EZ}}, I^{\text{EZ}})$ such that the time aggregator is isoelastic:[11]

$$W^{\text{EZ}}(c, y) = \begin{cases} \left((1-\beta)c^\rho + \beta y^\rho\right)^{\frac{1}{\rho}}, & \text{if } 0 \neq \rho < 1, \\ \exp\left((1-\beta)\log(c) + \beta \log(y)\right), & \text{if } \rho = 0, \end{cases}$$

and the certainty equivalent I^{EZ} is scale-invariant. The combination of isoelastic time aggregators and scale-invariant certainty equivalents implies that preferences are homothetic, which is of course extremely convenient in applications.[12] Building on Proposition 1, we can now characterize when such preferences are monotone.

To do so, note that a recursive representation $(U^{\text{EZ}}, W^{\text{EZ}}, I^{\text{EZ}})$ is equivalent to a representation $(\hat{U}, \hat{W}, \hat{I})$ such that $\hat{U} = f(U^{\text{EZ}})$, $\hat{W}(c, y) = f(W^{\text{EZ}}(c, f^{-1}(y)))$, and $\hat{I} = f I^{\text{EZ}} f^{-1}$, where f is a strictly increasing function. Using $f(x) = \frac{x^\rho}{\rho}$ in the case when $\rho \neq 0$ and $f(x) = \log(x)$ in the case $\rho = 0$, we obtain that the Epstein–Zin approach is equivalent to assuming

$$\begin{cases} \hat{W}(c, y) = (1-\beta)\dfrac{c^\rho}{\rho} + \beta y \text{ and } \hat{I}(\mu) = \dfrac{1}{\rho}\left(I^{\text{EZ}}\left((\rho\mu)^{\frac{1}{\rho}}\right)\right)^\rho, & \text{if } 0 \neq \rho < 1, \\ \hat{W}(c, y) = (1-\beta)\log(c) + \beta y \text{ and } \hat{I}(\mu) = \log\left(I^{\text{EZ}}\left(\exp(\mu)\right)\right), & \text{if } \rho = 0. \end{cases}$$

[11]The case where $\rho = 0$ is not explicitly written in Epstein and Zin (1989), but is obtained from the general case by taking the limit $\rho \to 0$.

[12]Preference homotheticity means that the ranking of two temporal lotteries is unaffected if multiplying all consumption levels by a positive scalar. In consumption-savings applications, this implies that the agent's wealth is just a simple scaling factor, which does not impact the agent's propensity to save and portfolio composition.

When $\rho = 0$, implying an elasticity of intertemporal substitution equal to 1, the scale invariance of I^{EZ} implies the translation invariance of $\hat{I}$, which, by Proposition 1, means that Epstein–Zin preferences are monotone. However, when $\rho \neq 0$, Monotonicity is only obtained when the certainty equivalent I^{EZ} satisfies $I^{\text{EZ}}(\mu) = (I(\mu^{\rho}))^{\frac{1}{\rho}}$ for some scale- and translation-invariant I.[13] We can, for example, consider a certainty equivalent I based on the dual model of Yaari (1987).[14]

The most popular specification of Epstein and Zin (1989) is actually obtained when I^{EZ} is of the expected utility form, $I^{\text{EZ}} = \phi^{-1} E\phi$, with ϕ given by

$$\phi(x) = \begin{cases} \frac{1}{\alpha} x^{\alpha}, & \text{if } 0 \neq \alpha < 1, \\ \log(x), & \text{if } \alpha = 0. \end{cases}$$

This also corresponds to the specification of Weil (1990), providing the so-called Epstein–Zin–Weil preferences. As mentioned above, Monotonicity is granted when $\rho = 0$, that is, when the elasticity of intertemporal substitution equals 1. When $\rho \neq 0$, Monotonicity is only obtained if $\mu \mapsto (E(\mu^{\frac{\alpha}{\rho}}))^{\frac{\rho}{\alpha}}$ is translation-invariant, implying $\alpha = \rho$. This corresponds to the standard additive expected utility model.[15] In all cases where $\rho \neq 0$ and $\alpha \neq \rho$, Epstein–Zin–Weil preferences are not monotone. We illustrate in Section 6 the implications of such a departure from Monotonicity.

4.4. *Attitudes Toward the Timing of Resolution of Uncertainty*

As emphasized in Kreps and Porteus (1978), recursive preferences may exhibit non-trivial attitudes toward the timing of resolution of uncertainty. In this section, we explain what a preference for early (or late) resolution of uncertainty implies for the representations of monotone recursive preferences we have derived. The analysis will also clarify the behavioral implications of having a scale-invariant certainty equivalent I, which is the main difference between the two cases obtained in Proposition 1.

In order to relate our paper to previous contributions discussing attitudes toward the timing of uncertainty, Chew and Epstein (1991) and Strzalecki (2013) in particular, we consider two notions of preference for early resolution of uncertainty. The first notion is directly taken from Chew and Epstein (1991).

DEFINITION 1: A binary relation $\succeq$ on D exhibits a preference for early resolution of uncertainty if, for all $n > 0$, $c_0, c_1 \in C$, $(m_i) \in (M(D))^n$, and $(\pi_i) \in [0,1]^n$ such that $\sum_{i=1}^{n} \pi_i = 1$, we have

$$A := \left(c_0, \bigoplus_{i=1}^{n} \pi_i(c_1, m_i)\right) \succeq \left(c_0, c_1, \bigoplus_{i=1}^{n} \pi_i m_i\right) =: B. \tag{9}$$

[13]For every $\rho \neq 0$ and $\mu \in M(\mathbb{R}_+)$, μ^{ρ} is the probability measure in $M(\mathbb{R}_+)$ such that $[\mu^{\rho}](B^{\rho}) = \mu(B)$ for every set $B \in \mathcal{B}(\mathbb{R}_+)$, where B^{ρ} denotes the set $\{y^{\rho} : y \in B\}$.

[14]This is equivalent to using the rank-dependent approach suggested by Epstein and Zin (1990), with the constraint that the parameters ρ and α they introduced in their paper have to be equal.

[15]The monotonicity of Epstein–Zin–Weil preferences when $\rho = 0$ or when $\rho = \alpha$ does not contradict Proposition 2, since in these cases Epstein–Zin–Weil preferences are also risk-sensitive preferences.

If the above ranking is one of indifference, then $\succeq$ exhibits indifference toward the timing of the resolution of uncertainty.[16]

The second notion obtains by restricting the lotteries $m_i \in M(D)$ in the above definition to be degenerate, that is, elements of D.

DEFINITION 2: A binary relation $\succeq$ on D exhibits a restricted preference for early resolution of uncertainty if, for all $n > 0$, $c_0, c_1 \in C$, $(c_{2i}, m_i) \in D$, and $(\pi_i) \in [0, 1]^n$ such that $\sum_{i=1}^n \pi_i = 1$, we have

$$A := \left(c_0, \bigoplus_{i=1}^{n} \pi_i(c_1, c_{2i}, m_i) \right) \succeq \left(c_0, c_1, \bigoplus_{i=1}^{n} \pi_i(c_{2i}, m_i) \right) =: B. \tag{10}$$

If the above ranking is one of indifference, then $\succeq$ exhibits restricted indifference toward the timing of the resolution of uncertainty.

This second definition is similar to that of Strzalecki (2013).[17] By construction, Definition 2 is weaker than Definition 1, which is emphasized by the adjective "restricted" that appears in Definition 2. In Definition 1, preference for early resolution of uncertainty means that the agent prefers when some uncertainty (but not necessarily all uncertainty) resolved in period 2 (in lottery B) is resolved in period 1 (in lottery A). In Definition 2 (restricted), preference for early resolution of uncertainty means that the agent prefers when all uncertainty resolved in period 2 (in lottery B) is resolved in period 1 (in lottery A). As stated in the next proposition, these two definitions yield different characterizations.

PROPOSITION 3: *Consider a monotone recursive preference relation $\succeq$ with a representation (U, W, I) as in Proposition 1. Then*:

– *$\succeq$ exhibits indifference toward the timing of resolution of uncertainty if and only if $I = E$*;

– *$\succeq$ exhibits restricted preference for early (resp. late) resolution of uncertainty resolution if and only if*:

∘ *either, $\succeq$ can be represented as in the first case of Proposition* 1, *with a certainty equivalent that in addition fulfills $I(\beta\mu) \geq \beta I(\mu)$ (resp. $I(\beta\mu) \leq \beta I(\mu)$) for all $\mu \in M([0, 1])$*;

∘ *or, $\succeq$ can be represented as in the second case of Proposition* 1.

– *if $\succeq$ is a Kreps–Porteus preference relation, restricted preference for early (resp. late) resolution of uncertainty is equivalent to unrestricted preference for early (resp. late) resolution of uncertainty, and also equivalent to having $k \geq 0$ (resp. $k \leq 0$) in the representation of risk-sensitive preferences given in Proposition 2.*

The proof is relegated to Section C of the Appendix. The first part of Proposition 3, which is due to Chew and Epstein (1991), shows that a separation of risk and intertemporal attitudes is possible only if the temporal resolution of uncertainty matters.[18] The

[16]Preference for late resolution of uncertainty is defined similarly, by reverting the ranking in (9), that is, by requiring that $B \succeq A$. The remark also applies to Definition 2 below.

[17]The definition of Strzalecki (2013) is written in a setting of subjective uncertainty. The parallel of Strzalecki's definition would be obtained when constraining the m_i that appear in equation (10) to be risk-free (i.e., elements of C^∞). However, with Recursivity, this would be equivalent to our definition.

[18]The result in Chew and Epstein (1991) is in fact stronger: it shows that Uzawa–Epstein preferences are the only recursive preferences that exhibit indifference toward the timing of resolution of uncertainty. In particular, recursive preferences that exhibit indifference to the timing of resolution of uncertainty are necessarily monotone.

significance of this point, which we discuss further in Section 7, has been recently emphasized by Epstein, Farhi, and Strzalecki (2014). The second part of Proposition 3 parallels Lemma 1 of Strzalecki (2013). It implies that scale-invariant certainty equivalents generate restricted indifference toward the timing of resolution of uncertainty. Thus, by choosing a scale-invariant certainty equivalent that is not of the expected utility form, it is possible to have a separation between risk and intertemporal attitudes while maintaining a restricted indifference toward the timing of resolution of uncertainty. A simple example is obtained by using certainty equivalents I based on the dual approach of Yaari (1987). The final part of Proposition 3 shows that this latter possibility, however, disappears if we further assume the independence axiom of Section 4.2.

5. INTUITION BEHIND PROPOSITION 1

This section provides insights behind our main result, Proposition 1. Maintaining Recursivity throughout, we focus on the distinct roles played by Monotonicity and Stationarity, with Stationarity taken in a broad sense that includes History Independence. While Stationarity (without Monotonicity) is known to imply the recursive representation shown in Lemma 1, we explain in Section 5.1 that assuming Monotonicity (without Stationarity) leads to a different type of recursive representation. Section 5.2 shows that combining these two recursive representations leads to a system of generalized distributivity equations, which is the starting point of the formal proof provided in Appendix B. Finally, in Section 5.3, we provide insights about the restrictions on risk preferences that arise when Monotonicity and Stationarity are combined.

5.1. *Implications of Monotonicity*

Consider a continuous preference relation $\succeq$ on D that fulfills Monotonicity. The relation $\succeq$ induces a preference relation on C^∞, which can be represented by a continuous function $V : C^\infty \to \text{Im}(V) \subset \mathbb{R}$.

For the sake of simplicity, we now focus on simple temporal lotteries whose uncertainty resolves in a single period $t \geq 1$. These are lotteries of the form

$$\left(c^t, \bigoplus_i \pi_i c_i^\infty\right), \tag{11}$$

for some $c^t \in C^t$, $c_i^\infty \in C^\infty$, and probabilities $\pi_i \in [0, 1]$ that sum to 1. A temporal lottery as in (11) can be associated with a lottery over lifetime utilities $\bigoplus_i \pi_i V(c^t, c_i^\infty) \in M^f(\text{Im}(V))$, where $M^f(\text{Im}(V))$ is the set of finite-support lotteries with outcomes in $\text{Im}(V)$. Moreover, because of Monotonicity, if two temporal lotteries as in (11) induce the same lottery in $M^f(\text{Im}(V))$, they have to be indifferent. Thus, the preference relation $\succeq$ on D induces a preference relation $\succeq_t$ on $M^f(\text{Im}(V))$.[19] Let $I_t : M^f(\text{Im}(V)) \to \text{Im}(V)$ be a continuous function representing $\succeq_t$ such that $I_t(x) = x$ for all $x \in \text{Im}(V)$. Monotonicity requires I_t to be increasing with respect to first-order stochastic dominance, which, in our terminology, means that I_t is a certainty equivalent. By construction, the function

$$\left(c^t, \bigoplus_i \pi_i c_i^\infty\right) \mapsto I_t\left(\bigoplus_i \pi_i V\left(c^t, c_i^\infty\right)\right) \tag{12}$$

[19]To be fully precise, for domain reasons, the preference relation $\succeq$ generates a preference relation $\succeq_t$ on a subset of $M^f(\text{Im}(V))$. This preference relation $\succeq_t$ can, however, be extended to the whole domain $M^f(\text{Im}(V))$. The formal proof of Proposition 1 addresses these technicalities.

affords a utility representation for the set of simple temporal lotteries that resolve in period t. Using Recursivity, we can extend this representation of preferences to the set of simple temporal lotteries. Indeed, a temporal lottery that resolves within the first T periods of time can be evaluated using the end point V_0 of the following backward recursion:

$$\begin{cases} V_t = V(c_0, c_1, \ldots) & \text{for } t = T, \\ V_t = I_{t+1}\big([V_{t+1}]\big) & \text{for all } t < T. \end{cases} \tag{13}$$

In words, a temporal lottery can be associated with a compound lottery over lifetime utilities, which is then evaluated recursively as in Segal (1990), using a sequence $I_1, I_2, \ldots$ of certainty equivalents. Note that the certainty equivalents I_t may depend on the date t. Such a dependence would generate non-trivial attitudes toward the timing of uncertainty (in both the restricted and unrestricted sense).

Note that the above recursion does not imply what Kreps and Porteus (1978) called the standard "pay-off vector approach" for evaluating temporal lotteries. The latter approach consists in first computing a compound lottery over lifetime utility, and then evaluating this lottery using the reduction of compound lottery axiom. The recursion in (13) requires one to preserve the first step of the pay-off vector approach, but not the latter, since the compound lottery over lifetime utility is evaluated recursively without reducing it to a one-stage lottery.

5.2. *A System of Generalized Distributivity Equations*

Assume now that the preference relation $\succeq$ fulfills both Monotonicity and Stationarity. From Monotonicity, we know that there exists a utility representation as in (13), while from Stationarity, we know that the preference relation also admits a recursive representation (U, W, I) as in Lemma 1. It turns out that the co-existence of the two representations is only possible in a few specific cases. To show this, we choose the lifetime utility V used for representation (13) to be the restriction of U to C^∞, which implies that $U = V_0$.[20] Applying representation (13) to lotteries that resolve in period 1, we obtain

$$W\bigg(c_1, I\bigg(\bigoplus_i \pi_i U\big(c_i^\infty\big)\bigg)\bigg) = I_1\bigg(\bigoplus_i \pi_i W\big(c_1, U\big(c_i^\infty\big)\big)\bigg), \tag{14}$$

and for lotteries that resolve in period 2:

$$W\bigg(c_0, W\bigg(c_1, I\bigg(\bigoplus_i \pi_i U\big(c_i^\infty\big)\bigg)\bigg)\bigg) = I_2\bigg(\bigoplus_i \pi_i W\big(c_0, W\big(c_1, U\big(c_i^\infty\big)\big)\big)\bigg). \tag{15}$$

Hence, substituting (14) into (15) and setting $y_i = W(c_1, U(c_i^\infty))$ yield

$$W\bigg(c_0, I_1\bigg(\bigoplus_i \pi_i y_i\bigg)\bigg) = I_2\bigg(\bigoplus_i \pi_i W(c_0, y_i)\bigg), \tag{16}$$

which is a functional equation that relates I_1, I_2, and W. An equation fully similar to (16), relating I_2, I_3, and W, can be obtained when considering temporal lotteries that resolve in

[20]Since U and V_0 represent the same preference relation on D and are identical on C^∞, they have to be identical.

periods 2 and 3. These two equations form a system of generalized distributivity equations which we solve in Appendix B using the work of Lundberg (1982, 1985).

Fulfilling this system of distributivity equations is found to impose non-trivial restrictions on ordinal preferences. We demonstrate that the time aggregator has to be affine up to a normalization, that is, to write as $W(c,x) = \phi^{-1}(u(c) + b(c)\phi(x))$ for some increasing (normalization) function ϕ. Only some stationary preferences over deterministic consumption paths can thus be extended to monotone recursive preferences over temporal lotteries.[21] Unfortunately, it seems difficult to come up with a straightforward intuition for this restriction on the time aggregator. The restrictions on the certainty equivalents that appear in Proposition 1 can, however, be made very intuitive, as we explain in the following section.

5.3. *Restrictions on Certainty Equivalents*

Let us take for granted that ordinal preferences admit a recursive representation with an affine time aggregator $W(c,x) = u(c) + b(c)x$ and consider a representation as in (13) with $V(c_0, c_1, \dots) = \sum_{i=0}^{\infty} u(c_i) \prod_{j=0}^{i-1} b(c_j)$ and monotone certainty equivalents I_t. Without further assumptions on the I_t, this yields preferences that fulfill Monotonicity but not necessarily Stationarity. Adding Stationarity as a requirement can be shown to provide very intuitive restrictions on the certainty equivalents I_t, with direct implications for the certainty equivalent I that appears in the recursive representation (U, W, I) used to formulate Proposition 1.

Consider indeed an agent comparing temporal lotteries that provide the same consumption profile $c^t = (c_0, \dots, c_{t-1})$ during the first $t > 0$ periods of time, but may differ thereafter. On the one hand, with Stationarity, the initial t periods of consumption do not matter and the ranking has to be independent of c^t. On the other hand, Monotonicity implies that future risks are evaluated in terms of their impact on lifetime utility, which depends on the utility derived from the t periods of initial consumption. In particular, take a lottery $(c'_0, \bigoplus_i \pi_i c_i^{\infty})$ that resolves after the first period. By continuity, there exists $c^{\infty} \in C^{\infty}$ such that $(c'_0, \bigoplus_i \pi_i c_i^{\infty}) \sim c^{\infty}$. By Stationarity, we can postpone the preceding lotteries by t periods and insert a consumption vector c^t without changing preferences. Namely, we have the indifference: $(c^t, c'_0, \bigoplus_i \pi_i c_i^{\infty}) \sim (c^t, c^{\infty})$. In terms of the representation (13), we obtain, for all $t > 0$, the implication

$$
\begin{aligned}
&I_1\Big(\bigoplus_i \pi_i V\big(c'_0, c_i^{\infty}\big)\Big) = V\big(c^{\infty}\big) \\
&\quad \Rightarrow \quad \Big(I_{t+1}\Big(\bigoplus_i \pi_i V\big(c^t, c'_0, c_i^{\infty}\big)\Big) = V\big(c^t, c^{\infty}\big), \forall c^t \in C^t\Big),
\end{aligned}
\tag{17}
$$

whose consequences are now investigated.

To begin, consider the case when the function b is constant ($b(c) = \beta \in (0,1)$ for all $c \in C$), implying that $V(\tilde{c}_0, \tilde{c}_1, \dots) = \sum_{i=0}^{\infty} \beta^i u(\tilde{c}_i)$. Then, inserting c^t, as in (17), impacts lifetime utility through an additive term, $\sum_{i=0}^{t-1} \beta^i u(c_i)$, and a factor β^t that multiplies

[21]More precisely, it is known that preferences that admit such a time aggregator comprise a much smaller class than the stationary preferences of Koopmans (1960): the former exhibit a strong form of impatience, which fails generically within the broader class. See Koopmans, Diamond, and Williamson (1964) and Epstein (1983) for details.

continuation utility. Changing c^t affects the additive term, shifting lifetime utility by a constant. For implication (17) to hold, the certainty equivalent I_t has to be translation-invariant. In other words, preferences must exhibit constant absolute risk aversion with respect to lifetime utility.

Inserting c^t, as in (17), has two other effects. The discount factor β^t, which multiplies continuation utility, scales down the risk on lifetime utility. Moreover, the resolution of uncertainty is postponed from period 1 till period $t+1$. Both effects may generate a breakdown of Stationarity, unless they are both separately neutralized or they cancel each other out. Formally, equation (17) holds if

$$I_{t+1}(\mu) = \beta^t I_1\left(\frac{1}{\beta^t}\mu\right). \tag{18}$$

One possibility is to have all I_t equal to a single uncertainty equivalent I fulfilling $I(\mu) = \beta I(\frac{1}{\beta}\mu)$. This will yield a representation exhibiting restricted indifference toward the timing of resolution of uncertainty. The other possibility is to allow the I_t to be different from each other, but related through (18). In that case, preference for the timing of resolution of uncertainty is used to generate an amplification mechanism that precisely compensates the decrease in risk due to the discount factor.

The risk-sensitive preferences of Hansen and Sargent (1995) provide an example in which Stationarity is preserved via the "amplification mechanism" shown in equation (18). In that case, $I_t = \phi_t^{-1} E \phi_t$ with $\phi_t = \frac{1-\exp(-k\beta^{-t}x)}{k\beta^{-t}}$ for some $k \in \mathbb{R} \setminus \{0\}$. Since $\beta < 1$, the coefficient of risk aversion $k\beta^{-t}$ increases with t if $k > 0$ and decreases with t if $k < 0$. This has direct consequences in terms of the agent's attitudes toward the timing of uncertainty. Indeed, in the case where $k > 0$, the sequence of certainty equivalents I_t is decreasing (in the sense that $I_{t+1}(\mu) \leq I_t(\mu)$ for all $\mu \in M(\mathbb{R}_+)$ and $t > 0$), implying a (restricted) preference for early resolution of uncertainty, while if $k < 0$, the sequence of certainty equivalents I_t is increasing, implying a (restricted) preference for late resolution of uncertainty. A relation therefore emerges between risk aversion and the agent's attitudes toward the timing of uncertainty, which is due to the necessity of compensating the scaling down of the risk generated by the discount factor.[22]

Suppose now that the function b is non-constant. In this case, the contribution of the first t periods to lifetime utility is slightly more complex, involving both an additive term $\sum_{i=0}^{t-1} u(c_i^i) \prod_{j=0}^{i-1} b(c_j)$ and a term $\prod_{j=0}^{t-1} b(c_j)$ that multiplies continuation utility. It is possible to change the consumption levels $c_0, \ldots, c_{t-1}$, so as to impact the additive term, without changing the multiplicative one.[23] As in the previous case, we can thus deduce that for the ranking of temporal lotteries to be independent of the consumption profile $(c_0, \ldots, c_{t-1})$, the certainty equivalents I_t have to be translation-invariant. More generally, changes in $(c_0, \ldots, c_{t-1})$ will affect the multiplicative term $\prod_{j=0}^{t-1} b(c_j)$ as well. For (17) to

[22] As was emphasized by Strzalecki (2013), multiplier preferences, which are the parallel of risk-sensitive preferences in a setting of subjective uncertainty, exhibit a similar relation between ambiguity aversion and preference for the timing of resolution of uncertainty.

[23] For example, one may consider permutations of the consumption levels during the first t periods: e.g., using $(c_1, c_0, c_2, \ldots, c_{t-1})$ instead of $(c_0, c_1, c_2, \ldots, c_{t-1})$.

hold, one must have

$$I_{t+1}(\mu) = \prod_{j=0}^{t-1} b(c_j) I_1\left(\frac{1}{\prod_{j=0}^{t-1} b(c_j)}\mu\right),$$

for all $(c_0, \dots, c_{t-1}) \in C^t$. The certainty equivalent I_1 must thus be scale-invariant, which implies that all the I_t are identical to I_1. The other option, which involved using different I_t with an amplification mechanism akin to the one of equation (18), has no analogue when b is non-constant.

Knowing that the time aggregator W is affine, the restrictions on the certainty equivalents I_t translate directly into restrictions on the certainty equivalent I used in recursion (6), which we employed in the formulation of our results. Indeed, in the case where b is constant, the certainty equivalent I is related to I_1 by $I(\mu) = \frac{1}{\beta} I_1(\beta\mu)$ and thus inherits the translation invariance of I_1. In the case where b is not constant, $I = I_1$ and I is both translation- and scale-invariant.

6. MONOTONICITY IN A CONSUMPTION-SAVINGS PROBLEM

This section illustrates the implications of Monotonicity in the context of a standard consumption-savings problem. One notable conclusion is that Monotonicity permits simple comparative statics regarding the role of risk aversion on the optimal level of saving. We consider a two-period economy. At date 0, the agent receives income y_0 which is certain and which she can allocate between consumption and savings. At date 1, one of two states, h or l, is realized. The states occur with probabilities $\pi_h \in (0, 1)$ and $\pi_l = 1 - \pi_h$, and determine both the income level at date 1, equal to y_1^h or y_1^l, and the gross return on savings, equal to R_h or R_l.

Throughout this section, we assume that preferences over deterministic consumption paths are represented by the function $U(c_0, c_1) = (c_0^\rho + \beta c_1^\rho)^{\frac{1}{\rho}}$, where $\beta > 0$ and $1 > \rho \neq 0$ is a parameter determining the elasticity of intertemporal substitution. Risk preferences will either be unspecified, though assumed to be monotone (for Lemma 2) or preferences à la Epstein–Zin–Weil (for Lemma 3 and Figure 1), or risk-sensitive (for the monotone case in Figure 1).

In the presence of uncertainty, the agent has to choose a level of savings before observing the state of the world. We denote by c_0^* the optimal consumption at date 0 and by s^* the optimal level of savings. The budget constraints are as follows:

$$y_0 - s^* = c_0^* \geq 0,$$
$$y_1^\kappa + R_\kappa s^* = c_{1,\kappa}^* \geq 0 \quad \text{for } \kappa = h, l,$$

where $c_{1,\kappa}^*$ denotes consumption at date 1 if state κ occurs. We use s_κ to denote the level of savings chosen if the agent had perfect foresight, that is, if she knew that state κ would occur for sure. We have

$$s_\kappa = \frac{y_0 - y_1^\kappa (\beta R_\kappa)^{\frac{1}{\rho-1}}}{1 + R_\kappa (\beta R_\kappa)^{\frac{1}{\rho-1}}} \quad \text{for } \kappa = h, l.$$

LEMMA 2—Savings With Monotone Preferences: *Consider the savings problem described above. If preferences are monotone, then* $s^* \in [\min(s_h, s_l), \max(s_h, s_l)]$.

PROOF: Assume that $s^* > \max(s_h, s_l)$, the case $s^* < \min(s_h, s_l)$ being completely symmetric. Since ordinal preferences are strictly convex, choosing $\hat{s} = \max(s_h, s_l)$ provides higher utility in both states of the world. This means that $(y_0 - \hat{s}, y_1^\kappa + R_\kappa \hat{s}) \succ (y_0 - s^*, y_1^\kappa + R_\kappa s^*)$ for both states $\kappa = h, l$, where $\succ$ denotes the strict preference relation. Then, Monotonicity implies

$$\left(y_0 - \hat{s}, \pi_l\left(y_1^l + R_l \hat{s}\right) \oplus \pi_h\left(y_1^h + R_h \hat{s}\right)\right) \succ \left(y_0 - s^*, \pi_l\left(y_1^l + R_l s^*\right) \oplus \pi_h\left(y_1^h + R_h s^*\right)\right),$$

which contradicts the optimality of s^*. *Q.E.D.*

The result of Lemma 3 reflects, therefore, that with monotone preferences, an agent would never choose a level of savings if another choice gives higher lifetime utility in both states of the world.

The result does not extend to non-monotone preferences. Indeed, assume that preferences can be represented by the function

$$U^{\mathrm{EZ}}(c_0, \tilde{c}_1) = \left(c_0^\rho + \beta\left(E\left[\tilde{c}_1^\alpha\right]\right)^{\frac{\rho}{\alpha}}\right)^{\frac{1}{\rho}}, \tag{19}$$

where $\alpha \neq 0$ is a parameter driving risk aversion, with larger α indicating lower risk aversion. Equation (20) is a representation of Epstein–Zin–Weil preferences, which are not monotone whenever $\rho \neq \alpha$, as discussed in Section 4.3. Let s^{EZ} be the optimal level of savings for an agent with such preferences, that is, let

$$s^{\mathrm{EZ}} = \arg \max_{s \in \left(-\min\left(\frac{y_1^l}{R_l}, \frac{y_1^h}{R_h}\right), y_0\right)} U^{\mathrm{EZ}}(y_0 - s, \tilde{y}_1 + \widetilde{R} s), \tag{20}$$

where $\tilde{y}_1$ and $\widetilde{R}$ denote the state-contingent income and asset returns.

LEMMA 3—Savings With Epstein–Zin–Weil Preferences: *Consider the savings problem described in equation* (20). *If $\rho \neq \alpha$, there exist values of R_κ and y_1^κ, $\kappa = h, l$, for which the agent chooses a level of savings $s^{\mathrm{EZ}} \notin [\min(s_h, s_l), \max(s_h, s_l)]$.*

PROOF: Assume $y_1^h \neq y_1^l$ and $R_\kappa = \frac{1}{\beta}(\frac{y_1^\kappa}{y_0})^{1-\rho}$ in state $\kappa = h, l$. In that case, $s_h = s_l = 0$, so that $[\min(s_h, s_l), \max(s_h, s_l)] = \{0\}$. However, we have

$$\frac{d}{ds}\left(\log U^{\mathrm{EZ}}(y_0 - s, \tilde{y}_1 + \widetilde{R} s)\right)\bigg|_{s=0} = \frac{y_0^{\rho-1}}{U^{\mathrm{EZ}}(y_0, \tilde{y}_1)}\left(\frac{E\left[\tilde{z}^{1-\frac{\rho}{\alpha}}\right]}{E[\tilde{z}]^{1-\frac{\rho}{\alpha}}} - 1\right), \tag{21}$$

where $\tilde{z} = \tilde{y}_1^\alpha$. Since $\rho \neq 0$ and $\rho \neq \alpha$, the function $x \mapsto x^{1-\frac{\rho}{\alpha}}$ is either strictly concave or strictly convex. Using Jensen inequality, the derivative (21) cannot be equal to zero. Thus $s^{\mathrm{EZ}} \neq 0$ and therefore $s^{\mathrm{EZ}} \notin [\min(s_h, s_l), \max(s_h, s_l)]$. *Q.E.D.*

To better understand the role of Monotonicity and the different conclusions of Lemmas 2 and 3, note that the proof of Lemma 3 builds on the particular case where the states h and l are such that, with perfect foresight, the saving decisions in both states would be identical, that is, $s_l = s_h$. The lifetime utilities in those states are, however, different. An agent, who lacks perfect foresight and has non-monotone preferences, may prefer to reduce the difference in lifetime utilities even if this reduces lifetime utility in both states. The saving decision s^{EZ} then responds to uncertainty and depends on the probabilities π_l

and π_h. In contrast, Monotonicity implies that the willingness to reduce risk, no matter how strong, cannot lead to a choice that reduces lifetime utility in all states of the world. In the special case when $s_l = s_h$, this also means that the agent's saving decision is unaffected by the uncertainty.

Building on the two-period example in this section, we may also emphasize that, because of the restrictions it imposes, Monotonicity affords an intuitive understanding of the role of risk aversion and simple comparative statics. Indeed, choice under uncertainty can then be seen as making a trade-off between state-specific utilities. If preferences are monotone and convex, the agent's optimal choice has to maximize a (possibly endogenous) convex combination of ex post lifetime utilities, just like a Pareto optimum has to maximize a convex combination of individual utilities. Risk aversion is then reflected in the weights that appear in this convex combination. In particular, stronger risk aversion requires that higher weights be assigned to the "bad states." Bommier, Chassagnon, and Le Grand (2012) formalized this reasoning and showed that, whenever Monotonicity is assumed, simple dominance arguments make it possible to derive general and intuitive conclusions about the role of risk aversion in many problems of interest.

A Precautionary Saving Example

To illustrate the last point, consider a simpler version of the above consumption-savings problem whereby only income is random with $y_1^h > y_1^l$. Since the asset return is the same in both states, we have $c_{1,h} > c_{1,l}$ whatever the agent's saving decision. One can thus regard state h as the "good state" and state l as the "bad state." Saving choices are such that $s_h < s_l$. With monotone preferences, the optimal saving choice has to lie in the interval $[s_h, s_l]$. Moreover, as was demonstrated in Bommier, Chassagnon, and Le Grand (2012), an increase in risk aversion involves selecting a level of savings that is closer to s_l, the best response in the bad state. Intuitively, in the presence of income uncertainty, saving provides an imperfect insurance device which is more intensively used when the degree of risk aversion increases. Non-monotone preferences may deliver different results: (i) the agent may choose to save more than she would if any of the states, including the worst one, were to occur for sure, and (ii) the role of risk aversion may be non-monotonic. Figure 1 illustrates the contrast between the saving patterns obtained with monotone and non-monotone preferences.[24]

7. DISCUSSION

Our contribution provides an insight into the difficult question about which preference specification to use when modeling dynamic choice. Of course, there is no simple take-home message, as the answer may surely depend on the context. We may, however, stress some pros and cons that emerge from our analysis. As we explain below, the choice of the domain is not innocuous either.

[24]The graphs are built using risk-sensitive preferences, $U^{\mathrm{RS}}(c_0, \tilde{c}_1) = c_0^\rho - \frac{\beta}{k}\log(E[e^{-k\tilde{c}_1^\rho}])$, and Epstein–Zin preferences (equation 19). We plot the optimal savings as a function of the risk aversion parameter, k or $-\alpha$. We use the following parameters: $\rho = \frac{1}{2}$, implying an intertemporal elasticity of substitution equal to 2, $R_l = R_h = \beta = 1$, $y_0 = 100$, $y_1^l = 100$, $y_1^h = 125$, and $\pi_h = 1 - \pi_l = 5\%$. In other words, the agent has a probability of 5% of earning a bonus equal to a quarter of the base wage in the next period. The amount of savings s^{EU} reported on the graphs corresponds to what is obtained with the standard additive model (i.e., when $\alpha = \rho$ or when $k = 0$). The optimal amount of saving in the good state, s_h, lies further below and is not reported for reasons of scale.

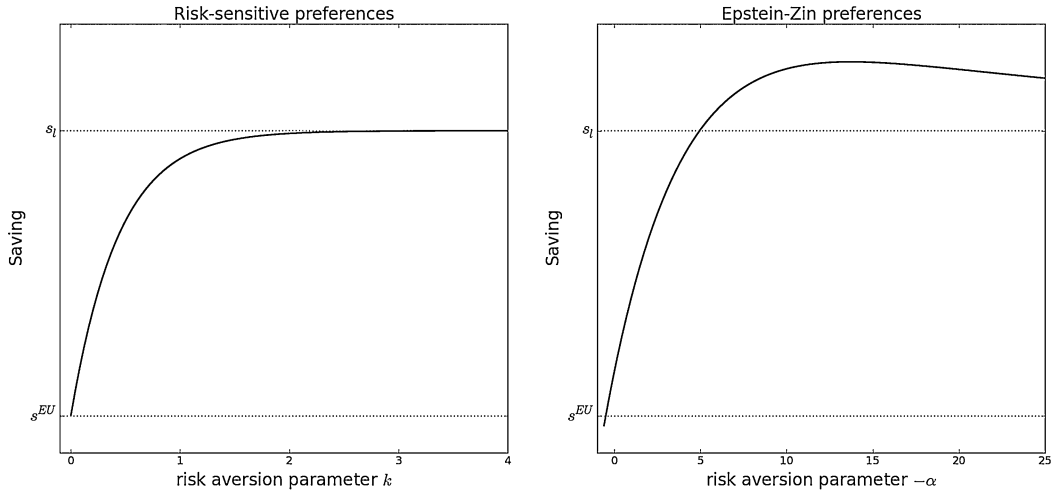

FIGURE 1.—The relation between risk aversion and savings.

First, the standard model of intertemporal choice, which is at the intersection of all models we mentioned, is well-behaved in all aspects and very tractable. Its main drawback, which is well known, is its lack of flexibility. Risk aversion is indeed fully determined by the properties of ordinal preferences.

Maintaining recursivity and stationarity leaves a few options to get flexibility. One is to impose monotonicity, which leads to the preferences studied in this paper. If we restrict the certainty equivalent to be of the expected utility form, then we arrive at the class of risk-sensitive preferences. A particular feature of risk-sensitive preferences is that they are not homothetic, unless the elasticity of substitution is equal to 1.[25] Homotheticity can be achieved by adopting a certainty equivalent I that is both translation- and scale-invariant, but is not of the expected utility form. Certainty equivalents based on the dual approach of Yaari (1987) fall into this category.

Another option is to depart from monotonicity. The most popular specification from Epstein and Zin (1989) and Weil (1990) (see Section 4.3) is homothetic and has a certainty equivalent of the expected utility form. These preferences have proved to be very tractable and have been used in a large number of studies. As we explain in Section 6, however, there are applications in which abandoning monotonicity may result in comparative statics that are difficult to interpret.

One can also gain flexibility, while preserving recursivity and monotonicity, by weakening stationarity. One possibility, which maintains history independence, is to use the recursion in (13) along with certainty equivalents I_t that are translation-invariant but not linked through the amplification mechanism of equation (18). The frameworks of Pye (1973) and van der Ploeg (1993) fall into this category. One can go further and allow for history dependence, while preserving sufficient structure so as to maintain reasonable tractability. A solution is to consider a time-additive function $V : C^\infty \to \mathbb{R}$ together with certainty equivalents of the form $I_t = \phi_t^{-1} E\phi_t$. Though generating some history dependence, the history can be summarized by a single variable, the stock of accumulated

[25]Homotheticity has well-known advantages. Recent contributions, however, have emphasized that departing from homotheticity may help explain some empirical regularities such as the relationship between trade flows and income per capita (Fieler (2011)) or between wealth and stock holdings (Wachter and Yogo (2010)).

welfare. Economic problems with such preferences can still be analyzed using standard dynamic programming techniques, with the introduction of only one additional state variable.[26]

The last point we want to make is that the choice of domain deserves careful consideration. Risk-sensitive preferences, which we showed to be the only class of monotone Kreps–Porteus recursive preferences affording enough flexibility to study risk aversion, impose a relationship between the agent's risk aversion and her attitudes toward the timing of resolution of uncertainty. As is explained in Section 5.3, this is a consequence of having a strictly positive rate of time preference, the latter being unavoidable in infinite horizon settings. The assumption of an infinite horizon setting, which was initially introduced by Koopmans (1960, p. 287) so as "to avoid complications connected with the advancing age and finite life span of the individual consumer," turns out to have far-reaching, possibly undesirable, consequences. The issue was already raised by Fisher who, in a comment that can be found in Koopmans (1965, pp. 298–300), argued that one should rather depart from the infinite horizon setting than accept the existence of time preferences.[27] Following the same line of arguments, one might want to abandon the infinite horizon setting so as to avoid the intertwinement of risk aversion and the agent's attitudes toward the timing of uncertainty. A possibility, pursued in Bommier (2013), is to restrict the domain, replacing the assumption of an infinite horizon by that of a possibly uncertain (but always finite) time horizon.

APPENDIX

This appendix is completed by the Supplemental Material (Bommier, Kochov, and Le Grand (2017)), containing technical details. All references to the Supplemental Material are prefixed with a letter S (e.g., Lemma S.15 or Section S.2).

APPENDIX A: MONOTONICITY IN IID AMBIGUITY MODELS

We derive here a result analogous to Proposition 1 in a stationary IID ambiguity setting similar to the one of Strzalecki (2013).[28] By stationary IID ambiguity we mean: (i) restricting the analysis to cases where the passing of time has no impact on the structure of the domain of choice; and (ii) introducing a set of assumptions implying that a decision maker who uses, at all dates, the same history independent preference relation is time-consistent. This is a restrictive approach, as it precludes the use of an arbitrary state space and rules out non-trivial belief updating. This framework has, however, proved very insightful in several instances. The exploration of more general settings is left for further work.[29] For mathematical rigor, we provide an axiomatic derivation using assumptions that parallel Axioms 1 to 5 of the main body of the paper. This axiomatization implies a recursive utility representation. The main contribution involves then showing that, like in the risk setting, significant restrictions are further obtained when imposing monotonicity.

[26] Bommier (2008) used such preferences to study life-cycle behavior, and assumed indifference toward the timing of the resolution of uncertainty (i.e., with functions ϕ_t independent of t).

[27] According to Fisher, "The obvious conclusion from Koopmans' paper, therefore, seems to me to be that one ought to abandon the use of infinite horizons—not that one ought to abandon certain ethical notions."

[28] The notion of IID ambiguity was first introduced in Epstein and Schneider (2003a) in the case of max–min expected utility representation.

[29] An investigation of the role of Monotonicity in the subjective uncertainty of Ju and Miao (2012) can also be found in Bommier and Le Grand (2014).

A.1. *Setup*

We consider a setup similar to that of Strzalecki (2013). Let S be a finite set representing the states of the world to be realized in each period. We assume that S has at least three elements and let $\Sigma := 2^S$ be the associated algebra of events. The full state space is $\Omega := S^\infty$, with a state $\omega \in \Omega$ specifying a complete history $(s_1, s_2, \ldots)$.[30] In each period $t > 0$, the individual knows the partial history $(s_1, \ldots, s_t)$. Such knowledge can be represented by a filtration $\mathcal{G} = (\mathcal{G}_t)_t$ on Ω where $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and, for every $t > 0$, $\mathcal{G}_t := \Sigma^t \times \{\emptyset, S\}^\infty$. We again let $C = [\underline{c}, \overline{c}]$ be the set of all possible consumption levels. A *consumption plan*, or an *act*, is a C-valued, $\mathcal{G}$-adapted stochastic process, that is, a sequence $h = (h_0, h_1, \ldots)$ such that $h_t : \Omega \to C$ is $\mathcal{G}_t$-measurable for every t. The set of all consumption plans is denoted by $\mathcal{H}$ and endowed with the topology of pointwise convergence.

We consider a binary relation $\succeq$ on $\mathcal{H}$ and introduce a set of axioms similar to those of Section 3. The axioms Weak Order, Continuity, and Monotonicity for Deterministic Prospects require no major modification. Below we state appropriate analogues for Axioms 3, 4, 5, and 7. Some notation is needed first. Given an act $h \in \mathcal{H}$ and state $\omega \in \Omega$, let $h(\omega) \in C^\infty$ be the deterministic consumption stream induced by h in state $\omega \in \Omega$, that is, $h(\omega) = (h_0, h_1(\omega), \ldots)$. Moreover, for any act $h \in \mathcal{H}$ and any $s \in S$, we define the *conditional act* $h^s \in \mathcal{H}$ by

$$\forall \omega = (s_1, s_2, \ldots) \in \Omega : h^s(s_1, s_2, \ldots) = h(s, s_2, \ldots) = \big(h_0, h_1(s, s_2, \ldots), \ldots\big). \tag{22}$$

The act h^s is obtained from h when knowing that the first component of the state of the world is equal to $s \in S$. Remark that $h^s(s_1, s_2, \ldots)$ is independent of s_1.

We can construct the *continuation act* $h^{s,1} \in \mathcal{H}$ from the conditional act h^s by removing the first-period consumption. Formally, for any act $h = (h_0, h_1, h_2, \ldots) \in \mathcal{H}$ and any $s \in S$, the continuation act $h^{s,1}$ is given by

$$\forall \omega = (s_1, s_2, \ldots) \in \Omega : h^{s,1}(s_1, s_2, \ldots) = \big(h_1(s, s_2, \ldots), h_2(s, s_2, \ldots), \ldots\big). \tag{23}$$

The continuation act $h^{s,1}$ can be viewed as the consumption plan implied by h starting at date 1 (ignoring date 0 consumption) and where the information revealed at the beginning of date 1 (i.e., s_1) is equal to s.

Last, for any $c \in C$ and $h \in \mathcal{H}$, we define the *concatenated act* $(c, h) \in \mathcal{H}$ by

$$(c, h) : \omega = (s_1, s_2, \ldots) \in \Omega \mapsto (c, h)(\omega) = \big(c, h(s_2, \ldots)\big) \in C^\infty. \tag{24}$$

The notions of conditional, continuation, and concatenated acts are related to each other. In particular, the conditional act is the concatenation of first-period consumption and the continuation act. Formally, for $h = (h_0, h_1, \ldots) \in \mathcal{H}$ and $s \in S$, we have $h^s = (h_0, h^{s,1})$. Moreover, any concatenated act (c, h) has continuation h. In mathematical terms, for any $c \in C$, $h \in \mathcal{H}$, and $s \in S$, we have $(c, h)^{s,1} = h$.

We can now state the axioms that parallel those given in the risk setting in Section 3.

[30]By setting $\Omega = S^\infty$ we constrain the state space to have a stationary structure (i.e., $\Omega = S \times \Omega$). If no such stationary structure were assumed, the passing of time would impact the structure of the preference domain. This means that (independently of time consistency issues) the same preferences could not be used at all dates. The domain of choice would simply change with time, which would require the use of different preference relations.

AXIOM A.3: *For all acts* $h = (h_0, h_1, \ldots)$ *and* $\hat{h} = (\hat{h}_0, \hat{h}_1, \ldots)$ *in* $\mathcal{H}$ *with* $h_0 = \hat{h}_0$,

$$(\forall s \in S, h^s \succeq \hat{h}^s) \quad \Rightarrow \quad h \succeq \hat{h}.$$

If, in addition, one of the former rankings is strict, then the latter ranking is strict.

Axiom A.3 is a concise statement that embeds both a property of recursivity and of state independence, the latter being implicit in the risk setting.[31] To be precise, recursivity alone would involve stating that for any $h, \hat{h} \in \mathcal{H}$ and $\sigma \in S$ such that $h \succeq \hat{h}$, and $h^s = \hat{h}^s$ for all $s \neq \sigma$,

$$(g \in \mathcal{H}, \hat{g} \in \mathcal{H}, g^\sigma = h^\sigma, \hat{g}^\sigma = \hat{h}^\sigma \text{ and } g^s = \hat{g}^s \text{ for all } s \neq \sigma) \quad \Rightarrow \quad g \succeq \hat{g}.$$

Such a property of recursivity makes it possible to combine time consistency and consequentialism in dynamic frameworks (see Johnsen and Donaldson (1985)). State independence extends the requirement of having $g \succeq \hat{g}$ to cases where there exists a state of the world $\hat{\sigma} \in S$ (possibly different from σ) such that $g^{\hat{\sigma}} = h^\sigma$, $\hat{g}^{\hat{\sigma}} = \hat{h}^\sigma$, and $g^s = \hat{g}^s$ for all $s \neq \hat{\sigma}$. When plugged into a dynamic framework, the state independence property translates into a form of history independence, in the sense that preferences regarding the future have to be independent of which states realized in past periods. Many papers relax the state-independent assumption allowing for non-trivial updating of beliefs. A prominent example is Hayashi (2005), who provided axiomatic foundations for more general recursive preferences, in a more complex setting that combines both objective and subjective uncertainty. As already mentioned, we leave for further work the exploration of the consequences of assuming Monotonicity in such more general settings.

Axiom 4 (History Independence) rewrites as follows:

AXIOM A.4: *For all acts* $h = (h_0, h_1, \ldots)$ *and* $\hat{h} = (h_0, \hat{h}_1, \ldots)$ *in* $\mathcal{H}$, *and* $\hat{h}_0 \in C$,

$$(h_0, h_1, \ldots) \succeq (h_0, \hat{h}_1, \ldots) \quad \Leftrightarrow \quad (\hat{h}_0, h_1, \ldots) \succeq (\hat{h}_0, \hat{h}_1, \ldots).$$

Regarding stationarity, Axiom 5 becomes the following:

AXIOM A.5: *For all* $c \in C$ *and* $h, \hat{h} \in \mathcal{H}$, *we have* $(c, h) \succeq (c, \hat{h}) \Leftrightarrow h \succeq \hat{h}$.

This assumption states that the comparison of two acts that assume the same deterministic consumption in period 0 and whose continuation acts are independent of the information revealed in the first period, can be done by comparing their respective continuation acts (with the same preference relationship $\succeq$).

To avoid confusion, we shall emphasize that our stationary assumption differs from that of Kochov (2015). In Kochov's paper, the information tree has an arbitrary exogenous structure, which does not allow him to define stationarity in the same way as we do. Kochov's stationarity is a property of preference invariance when changing the timing of consumption, while holding fixed the timing of resolution of uncertainty. In contradistinction, our stationarity assumption (Axiom A.5) is a property of preference invariance when

[31]In the risk setting, state independence is readily imposed by the fact that preferences are defined over lotteries, and not over random variables.

changing both the timing of consumption and the timing of resolution of uncertainty. Indeed, for a given $c \in C$ and a given $h \in \mathcal{H}$, the concatenated act (c, h), as defined in equation (24) is obtained by adding one initial period consumption c and postponing the timing of resolution of uncertainty by one period. For example, if h only depends on information revealed in the first period, then (c, h) only depends on the information revealed in the second period. With respect to the mathematical formalism, the difference between Kochov's approach and ours lies in the way the concatenation operation is defined.[32] This eventually leads to assumptions of different nature, unless the agent exhibits indifference to the timing of uncertainty resolution.

Central to our analysis is the assumption of Monotonicity:

AXIOM A.7—Monotonicity: *For any h and $\hat{h}$ in $\mathcal{H}$,*

$$\big(h(\omega) \succeq \hat{h}(\omega) \text{ for all } \omega \in \Omega\big) \quad \Rightarrow \quad h \succeq \hat{h}. \tag{25}$$

The above monotonicity axiom can be found in Epstein and Schneider (2003b), Maccheroni, Marinacci, and Rustichini (2006), and Kochov (2015). Note that this axiom is "stronger" than the one we used in the risk setting. An exact analogue of the risk axiom would restrict h and g to depend on the uncertainty resolving in a single period only. As the analysis in the risk setting suggests, the representation result we state in Proposition 4 below would continue to hold even if we were to weaken Axiom A.7 accordingly. We adopt Axiom A.7 because it is standard in the literature on subjective uncertainty and because we want to emphasize that the preferences we consider are in fact monotone in the strong sense of Axiom A.7.[33]

As in the risk setting, we say that a binary relation $\succeq$ on $\mathcal{H}$ is a *monotone recursive preference relation* if it satisfies Axioms 1, 2, A.3, A.4, A.5, 6, and A.7.

A.2. *Representation Result*

Let $B_0(\Sigma)$ be the set of simple, Σ-measurable functions from S into $\mathbb{R}_+$. The next definitions parallel those in Section 4.1. A *certainty equivalent* $I : B_0(\Sigma) \to \mathbb{R}_+$ is a continuous, strictly increasing function such that $I(x) = x$ for any $x \in \mathbb{R}_+$. A certainty equivalent is *translation-invariant* if for all $x \in \mathbb{R}_+$ and $f \in B_0(\Sigma)$, $I(x + f) = x + I(f)$. It is *scale-invariant* if for all $\lambda \in \mathbb{R}_+$ and $f \in B_0(\Sigma)$, $I(\lambda f) = \lambda I(f)$. Given a function $U : \mathcal{H} \to \mathbb{R}$ and an act $h \in \mathcal{H}$, we let $U \circ h^1$ denote the function $s \in S \mapsto U(h^{s,1})$. If U is a utility function, then $U \circ h^1$ is the state-contingent profile of continuation utilities induced by the act h in period 1. Letting $W : C \times [0, 1] \to [0, 1]$ be a time aggregator, a *recursive representation* for $\succeq$ is a tuple (U, W, I) such that $U : \mathcal{H} \to \mathbb{R}$ represents $\succeq$ and satisfies the recursion

$$U(h) = W\big(h_0, I\big(U \circ h^1\big)\big), \quad \text{where } U \circ h^1 : s \in S \mapsto U\big(h^{s,1}\big). \tag{26}$$

It is relatively simple to show that Axioms 1, 2, A.3, A.4, and A.5 are necessary and sufficient conditions for preferences to have a recursive representation (Lemma S.15 in Section S.2). Our contribution involves showing that further restrictions on the recursive representation appear when assuming preference monotonicity.

[32] In Kochov (2015), the concatenation (c, h) is defined by $(c, h)(s_1, s_2, \ldots) = (c, h(s_1, s_2, \ldots))$, which differs from the definition introduced in equation (24).

[33] As already mentioned, the axiom employed in the risk setting can be strengthened so as to obtain an analogue of Axiom A.7. The appropriate formulation is provided in Bommier and Le Grand (2014).

PROPOSITION 4: *A binary relation $\succeq$ on $\mathcal{H}$ is a monotone recursive preference relation if and only if it admits a recursive representation (U, W, I) such that either*:

1. *I is translation-invariant and $W(c, x) = u(c) + \beta x$ satisfies the conditions listed in the first case of Proposition 1*, *or*
2. *I is translation- and scale-invariant and $W(c, x) = u(c) + b(c)x$ satisfies the conditions listed in the second case of Proposition 1*.

This proposition parallels Proposition 1 obtained in the risk setting. Its proof can be found in Section S.2 of the Supplemental Material.

APPENDIX B: PROOF OF PROPOSITION 1

Necessity of the axioms is obvious. Proving sufficiency is a long task, but a good account of the proof can be found in Sections B.1, B.2, and B.3. To save space, we omit some technical details in these sections. These details can be found in the long version of the proof provided in the Supplemental Material. The core of the proof involves solving a system of *generalized distributivity equations*.

A generalized distributivity equation is an equation of the form

$$F\big(c, G(x_1, \dots x_m)\big) = H\big(F(c, x_1), \dots, F(c, x_m)\big), \tag{27}$$

where F, G, and H are continuous and strictly increasing functions, with F defined on some subset of $\mathbb{R}^2$ and G, H on subsets of $\mathbb{R}^m$. Such equations were studied in the case when $m = 2$ in Aczél (1966, Chapter 7) and under the assumption that the solutions are twice continuously differentiable. Much later, Lundberg (1982, 1985) addressed the non-differentiable case. In Section B.1, we explain Lundberg's approach and how it can be adapted to our problem. The formal proof of our result follows in Sections B.2 to B.7. Before entering into more details, we introduce some notation.

Notation

The composition of two functions f and g, when it is well-defined, is denoted as fg. Given an integer $n \geq 0$ and a function $f : X \to X$, f^n denotes the nth iterate of the function f. Thus, for example, f^2 stands for the function ff. In what follows, we often work with an ambient space X and real-valued functions f, f' that are defined on proper subsets of X. When we write $f(x)$, it is implicitly understood that x lies in the domain of f. Similarly, when we write $f > f'$, the expression is understood to hold for $x \in X$ such that f, f' are both defined.

B.1. *Insights From Lundberg (1982, 1985)*

Lundberg (1982, 1985) studied the equation (27) in the case where $m = 2$ and the functions H, F, and G are defined on rectangular domains. His strategy can be decomposed into three steps: (1) transforming the generalized distributivity equation with three unknown functions into a standard distributivity equation with two unknown functions; (2) deriving a linear distributivity equation; (3) solving the linear distributivity equation by showing that its solutions are necessarily differentiable and then using differential calculus to compute the solutions. The crucial point is the second one, which uses a fundamental result that relates iteration groups and Abel functions. This section provides a brief account of Lundberg's approach without aiming at full rigor. In particular—and in this section only—we are voluntarily evasive about domain issues, which are in fact crucial in the analysis. This allows us to skip many technicalities. All formal aspects are, however, carefully exposed when we derive our own proof (Sections B.2 to B.7).

Transforming a Generalized Distributivity Equation Into a Standard Distributivity Equation.

As in Lundberg (1982, 1985), we restrict ourselves to the case where $m = 2$, so that equation (27) can be rewritten as

$$F\big(c, G(x_1, x_2)\big) = H\big(F(c, x_1), F(c, x_2)\big). \tag{28}$$

To transform the generalized distribution equation (28) into a standard distributivity equation, where G and H are constrained to be equal, we fix some c_0 and define a function $(c, x) \mapsto f(c, x)$ by $F(c_0, f(c, x)) = F(c, x)$. The function f can be shown to exist on a restricted but non-trivial domain. Equality (28) applied with $c = c_0$ and $(x'_1, x'_2) = (f(c, x_1), f(c, x_2))$ yields

$$F\big(c_0, G\big(f(c, x_1), f(c, x_2)\big)\big) = H\big(F\big(c_0, f(c, x_1)\big), F\big(c_0, f(c, x_2)\big)\big).$$

Using the definition of f and equation (28), we obtain $F(c_0, G(f(c, x_1), f(c, x_2))) = H(F(c_0, f(c, x_1)), F(c_0, f(c, x_2)))$ and

$$f\big(c, G(x_1, x_2)\big) = G\big(f(c, x_1), f(c, x_2)\big), \tag{29}$$

which is a standard distributivity equation (note that the domains of equations (28) and (29) generally differ).

Deriving a Linear Distributivity Equation

For any c, we define the function f_c by $f_c(z) := f(c, z)$. Let us now fix some c_1. Equation (29), taken with $c = c_1$, provides

$$f_{c_1}\big(G(x_1, x_2)\big) = G\big(f_{c_1}(x_1), f_{c_1}(x_2)\big). \tag{30}$$

Lundberg's approach to solving the latter equation is based on the notion of a (continuous) iteration group. We start by providing some intuition for this notion and how it relates to the equation at hand. First, one can show that $f_{c_1}^{-1}$, the inverse of f_{c_1} (when defined), is also a solution of (30). One can then prove by induction that for any integer n, the iterates $f_{c_1}^n$ and $f_{c_1}^{-n}$ solve (30) as well. The idea of an iteration group is to extend the notion of a function iterate $f_{c_1}^\alpha$ to non-integer α. Namely, the iterates $(f_{c_1}^n)_{n\in\mathbb{Z}}$, which solve (30), can be extended to a family $\{f_{c_1}^\alpha\}_\alpha$ of functions which solve (30) as well and where α varies continuously in some open interval $(-\lambda, \lambda)$. The elements of $\{f_{c_1}^\alpha\}_{\alpha\in(-\lambda,\lambda)}$ "iterate" in the sense that $f_{c_1}^{\alpha+\alpha'} = f_{c_1}^\alpha f_{c_1}^{\alpha'}$ and $(f_{c_1}^\alpha)^{-1} = f_{c_1}^{-\alpha}$. The family $\{f_{c_1}^\alpha\}_{\alpha\in(-\lambda,\lambda)}$, denoted $\{f_{c_1}^\alpha\}_\alpha$ when no confusion arises, is called an *iteration group*.

A fundamental property of iteration groups is that any fixed-point-free iteration group admits an *Abel function*. In words, there exists a continuous and strictly increasing function L (the so-called *Abel function*) such that, for all $\alpha \in (-\lambda, \lambda)$, $f_{c_1}^\alpha(x_1) = L^{-1}(\alpha + L(x_1))$ (see Lundberg (1982, p. 79), for more details about Abel functions).[34] Equation (30) becomes

[34]This is a fundamental result that traces back to the solution of the translation equation (see Aczél (1966), Section 1 in Chapter 6).

$L^{-1}(\alpha+LG(x_1,x_2)) = G(L^{-1}(\alpha+L(x_1)), L^{-1}(\alpha+L(x_2)))$, or after defining $\widetilde{G}(x_1,x_2) := LG(L^{-1}(x_1), L^{-1}(x_2))$ and $y_1 := L(x_1)$ and $y_2 := L(x_2)$,

$$\alpha + \widetilde{G}(y_1, y_2) = \widetilde{G}(\alpha + y_1, \alpha + y_2).$$

Therefore, $\widetilde{G}$ is translation-invariant, which makes it possible to define a function ϕ by setting $\phi(y_2 - y_1) := \widetilde{G}(y_1, y_2) - y_1$. Denoting $\tilde{f}_c := Lf_cL^{-1}$, equation (30) rewrites as $\tilde{f}_c(\widetilde{G}(y_1, y_2)) = \widetilde{G}(\tilde{f}_c(y_1), \tilde{f}_c(y_2))$ or, using the definition of ϕ, as

$$\tilde{f}_c\big(x_1 + \phi(y_2 - y_1)\big) = \phi\big(\tilde{f}_c(y_2) - \tilde{f}_c(y_1)\big) + \tilde{f}_c(y_1), \tag{31}$$

which is a *linear distributivity equation*.

Differential Methods to Solve Linear Distributivity Equations

Lundberg (1985) studied linear distributivity equations of the form

$$f\big(x_1 + \phi(y_2 - y_1)\big) = \psi\big(f(y_2) - f(y_1)\big) + f(y_1), \tag{32}$$

which is slightly more general than (31), and corresponds to the linearization of a generalized distributivity equation as in (28). Lundberg showed that any solution of (32) is necessarily continuously differentiable, and then used differential calculus to compute solutions, an explicit account of which can be found in Lundberg (1985).

B.2. *Road Map for our Proof*

The first part of the proof shows that combining Monotonicity and Stationarity leads to a system of generalized distribution equations. The intuition was already given in Section 5.3, but Section B.3 provides a precise formal derivation. The problem we then need to solve differs from the one addressed by Lundberg in three respects. First, we do not restrict ourselves to the case when $m = 2$ (see equation (32)). Second, we have to solve a system of generalized distributivity equations that are related to each other, and not a single generalized distributivity equation. Last, these generalized distributivity equations hold on domains that are not rectangular, in contrast to Lundberg's original work. These aspects, and in particular the domain issues, forbid a direct application of Lundberg's results, even though the general strategy remains valid. The remainder of the proof is organized as follows. (1) In Sections B.4 to B.6, we construct an iteration group that allows us to transform our system of generalized distributivity equations into a system of linear distributivity equations. (2) Once these linear distributivity equations are derived, the results of Lundberg (1985) can be applied. Because we have a system of distributivity equations, where equations are related to each other, we can show that there remain only two possible solutions. These solutions provide the two cases of Proposition 1. (3) The final part of the proof in Section B.7 elicits the restrictions imposed by Axiom 6.

B.3. *Deriving a System of Distributivity Equations*

From Lemma 1, the preference relation $\succeq$ has a recursive representation (U, W, I). It is w.l.o.g. to assume that $U(D) = [0, 1]$. Fix some integer $m > 2$. Let $\mathcal{W}_0 := [0, 1]^m$ and

$$\mathcal{W}_1 := \big\{\big(W(c, x_1), \dots, W(c, x_m)\big) : c \in C, (x_1, \dots, x_m) \in \mathcal{W}_0\big\},$$

$$\mathcal{W}_2 := \big\{\big(W(c, x_1), \dots, W(c, x_m)\big) : c \in C, (x_1, \dots, x_m) \in \mathcal{W}_1\big\}.$$

Note that $\mathcal{W}_0 \supset \mathcal{W}_1 \supset \mathcal{W}_2$.

Now fix $(\pi_1, \dots, \pi_m) \in (0,1)^m$ such that $\sum_i \pi_i = 1$. For every $(x_1, \dots, x_m) \in [0,1]^m$, let $(\pi_1, x_1; \dots; \pi_m, x_m)$ be the lottery in $M([0,1])$ that gives x_k with probability π_k. Define a function $G_0 : \mathcal{W}_0 \to [0,1]$ by

$$G_0(x_1, \dots, x_m) := I\big((\pi_1, x_1; \dots; \pi_m, x_m)\big), \quad \forall (x_1, \dots, x_m) \in [0,1]^m, \tag{33}$$

which is the certainty equivalent of the lottery $(\pi_1, x_1; \dots; \pi_m, x_m)$. For $k \in \{1,2\}$, define a function $G_k : \mathcal{W}_k \to [0,1]$ inductively by letting

$$G_{k+1}\big(W(c, x_1), \dots, W(c, x_m)\big) := W\big(c, G_k(x_1, \dots, x_m)\big). \tag{34}$$

The functions G_k, $k \in \{1,2\}$, are well-defined by Monotonicity. For every $c \in C$, let F_c denote the function $x \mapsto W(c,x)$ from $[0,1]$ into $[0,1]$. Each function F_c is continuous and strictly increasing. Then, equation (34) becomes, for $k = 1, 2$,

$$\begin{cases} G_1\big(F_c(x_1), \dots, F_c(x_m)\big) = F_c G_0(x_1, \dots, x_m), & c \in C, (x_1, \dots, x_m) \in \mathcal{W}_1, \\ G_2\big(F_c(x_1), \dots, F_c(x_m)\big) = F_c G_1(x_1, \dots, x_m), & c \in C, (x_1, \dots, x_m) \in \mathcal{W}_2, \end{cases} \tag{35}$$

which is a system of generalized distributivity equations. The two equations in (35) are related through the function G_1, which appears in both.

Following the approach of Section B.1, we can derive standard distributivity equations from the generalized distributivity equations in (35). To simplify our notation, let $\beta := W(\underline{c}, 1)$. If $F_{\overline{c}}(0) > \beta$, let c^* be such that $F_{c^*}(0) = \beta$. Alternatively, if $F_{\overline{c}}(0) \leq \beta$, let $c^* := \overline{c}$. In each case, we have $F_c^{-1}[0, \beta] = [0, F_c^{-1}(\beta)] \neq \emptyset$ for every $c < c^*$. Take $c < c^*$, $k \in \{0, 1\}$, and $(x_1, \dots, x_m) \in [0, F_c^{-1}(\beta)]^m \cap \mathcal{W}_k$. Applying $F_{\underline{c}}^{-1}$ to both sides of (35) implies $F_{\underline{c}}^{-1} G_{k+1}(F_c(x_1), \dots, F_c(x_m)) = F_{\underline{c}}^{-1} F_c G_k(x_1, \dots, x_m)$. Combining this equation for an arbitrary c with the same equation for $c = \underline{c}$ yields

$$G_k\big(F_{\underline{c}}^{-1} F_c(x_1), \dots, F_{\underline{c}}^{-1} F_c(x_m)\big) = F_{\underline{c}}^{-1} F_c G_k(x_1, \dots, x_m). \tag{36}$$

Defining $f_c := F_{\underline{c}}^{-1} F_c$, (36) becomes, for $c < c^*$, $(x_1, \dots, x_m) \in [0, F_c^{-1}(\beta)]^m \cap \mathcal{W}_k$,

$$G_k\big(f_c(x_1), \dots, f_c(x_m)\big) = f_c G_k(x_1, \dots, x_m), \quad k = 0, 1, \tag{37}$$

which are distributivity equations similar to (29).

B.4. *Constructing an Iteration Group*

This part requires some mathematical machinery from Lundberg (1982). Given a proper interval $A \subset \mathbb{R}$, let $\mathcal{D}(A)$ be the set of all continuous, strictly increasing functions f whose domain and range are intervals contained in A and whose graphs disconnect A^2. The notion of iteration group was already introduced in Section B.1. Here, when we specify an iteration group $\{f^\alpha\} \subset \mathcal{D}(A)$, we assume that the group is *non-trivial*, that is, that $f^\alpha \neq f^0$ (f^0 is the identity function on A) for at least one $\alpha \neq 0$. If the group is non-trivial, then $f^\alpha \neq f^0$ for all $\alpha \neq 0$. Furthermore, when we specify an iteration group $\{f^\alpha\}_{\alpha \in (-\lambda, \lambda)}$ on a bounded interval A, we assume that the group is *maximal*, that is, there is no other iteration group $\{g^\alpha\}_{\alpha \in (-\lambda', \lambda')} \subset \mathcal{D}(A)$ such that $\lambda' > \lambda$ and $g^\alpha = f^\alpha$ for all $\alpha \in (-\lambda, \lambda)$.

Let $(f_n)_n$ be a sequence of functions in $\mathcal{D}(A)$. A function $f \in \mathcal{D}(A)$ is the *closed limit of* $(f_n)_n$, which we denote as $f_n \to_L f$, if the graph of f is the closed limit of the graphs of the functions f_n.[35] If A is a closed interval and the graphs of f_n and f are closed, then $f_n \to_L f$ if and only if the graphs of f_n converge to the graph of f in the Hausdorff metric. We write $f_n \to_H f$ to denote the latter type of convergence. The sequence $(f_n)_n$, $f_n \in \mathcal{D}(A)$, *generates the iteration group* $\{f^\alpha\}$ on A if for every $\alpha \in (-\lambda, \lambda)$, there exists a sequence $(p_n)_n$ of integers such that $f_n^{p_n} \to_L f^\alpha$.

We come back to the proof of the theorem. Let j be the identity function on [0, 1]. Fix a sequence $(c_n)_n$ such that $c_n \in (\underline{c}, c^*)$ for every n and the sequence decreases monotonically to $\underline{c}$. Let $(f_{c_n})_n$ be the associated sequence of functions where $f_{c_n} = F_{\underline{c}}^{-1} F_{c_n}$ for every n. We note several properties of the sequence $(f_{c_n})_n$. First, $f_{c_n} > f_{c_{n+1}} > j$ for every n. Second, each function f_{c_n} has domain $\mathrm{Dom}_n := [0, F_{c_n}^{-1}(\beta)]$ and range $[f_{c_n}(0), 1]$. It follows that the graph of each function f_{c_n} disconnects $[0, 1]^2$ so that $f_{c_n} \in \mathcal{D}([0, 1])$. Another immediate implication is that $\mathrm{Dom}_n \to_H [0, 1]$, which implies that for every $x \in (0, 1)$, there is $k > 0$ such that $f_{c_n}(x)$ is defined for all $n \geq k$. The sequence $(f_{c_k}(x), f_{c_{k+1}}(x), \dots)$ converges to x and the convergence can be shown to be uniform (see Lemma S.1 in the Supplemental Material).

The next two lemmas are key for solving the distributivity equation.

LEMMA 4—Constructing an Iteration Group: *There exists an iteration group* $\{f^\alpha\}_{\alpha \in (-\lambda, \lambda)}$ *on* $(0, 1)$ *such that* $\lambda > 1$, $f^\alpha > j$ *for all* $\alpha > 0$, *and*

$$f^\alpha G_0(x_1, \dots, x_m) = G_0\big(f^\alpha(x_1), \dots, f^\alpha(x_m)\big), \tag{38}$$

for all $(x_1, \dots, x_m) \in [0, 1]^m$ *and* $\alpha \in (-\lambda, \lambda)$ *for which the equation is well-defined.*

PROOF: We know that $f_{c_n} \to_L j$, $f_{c_n} \neq j$ for every n, and $\mathrm{Dom}_n \to_L (0, 1)$. Theorem 4.16 in Lundberg (1982) shows that $(f_{c_n})_n$ has a subsequence that generates the desired iteration group.[36] Abusing notation, from now on we write $(f_{c_n})_n$ for the latter subsequence. *Q.E.D.*

LEMMA 5—Constructing an Abel Function: *There is a continuous, strictly increasing function* $L : (0, 1) \to \mathbb{R}$ *such that* $f^\alpha(x) = L^{-1}(L(x) + \alpha)$ *for all* x *in the domain of* f^α *and all* $\alpha \in (-\lambda, \lambda)$.

PROOF: We know that $f^\alpha > j$ for all $\alpha \in (0, \lambda)$. Since f^α is the inverse of $f^{-\alpha}$, the latter implies that $f^\alpha < j$ for all $\alpha \in (-\lambda, 0)$. In particular, none of the functions f^α, $\alpha \neq 0$, has a fixed point. As explained in Section B.1, the iteration group then has an Abel function, that is, a continuous function $L : (0, 1) \to \mathbb{R}$ such that $f^\alpha(x) = L^{-1}(\alpha + L(x))$ for every $\alpha \in (-\lambda, \lambda)$ and every x in the domain of f^α. Since $f^\alpha > j$ for all $\alpha > 0$, the function L is strictly increasing. *Q.E.D.*

Recall that each function f_{c_n} is defined in a right neighborhood of 0, while its range includes a left neighborhood of 1. It follows that for each $\alpha > 0$, $f^\alpha(0) := \lim_{x \searrow 0} f^\alpha(x)$ and for each $\alpha < 0$, $f^\alpha(1) := \lim_{x \nearrow 1} f^\alpha(x)$ are well-defined. From now on, we assume that

[35] See Aliprantis and Border (1999, p. 109) for the definition of a closed limit.

[36] The statement of Theorem 4.16 in in Lundberg (1982) does not say that $\lambda > 1$ and $f^\alpha > j$ for all $\alpha > 0$, but these properties of the iteration group follow from the proof of the theorem and the fact that $f_{c_n} > j$ for every n.

$\{f^\alpha\}$ is such that $f^1(0) > 0$ and $f^{-1}(1) < 1$.[37] Under this assumption, we have $f^\alpha(0) > 0$ for all $\alpha > 0$ and $f^\alpha(1) < 1$ for all $\alpha < 0$ as well as $L(0) := \lim_{x\searrow 0} L(x) > -\infty$ and $L(1) := \lim_{x\nearrow 1} L(x) < +\infty$. Using the latter, we now argue that the Abel function L can be chosen so that $L(0) = 0$ and $L(1) = 1$. First, since L is defined up to a translation, we can choose L so that $L(0) = 0$. To see that L can be chosen so that $L(1) = 1$, observe that $\lambda = \lim_{\alpha\nearrow\lambda} f^\alpha(0) = L(1)$. Relabeling the iteration group $\{f^\alpha\}_{\alpha\in(-\lambda,\lambda)}$ so that $\lambda = 1$ implies that $L(1) = 1$.

B.5. *A Monotone Transformation of Utility*

Since $L : [0, 1] \to [0, 1]$ is strictly increasing, the function $\widetilde{U} := LU : D \to [0, 1]$ represents $\succeq$ on D. Moreover, the function $\widetilde{U}$ is part of a recursive representation $(\widetilde{U}, \widetilde{W}, \widetilde{I})$ where

$$\widetilde{W}(c, x) := LW\big(c, L^{-1}(x)\big) \quad \forall x \in [0, 1], c \in C,$$
$$\widetilde{I}(\mu) := LI\big(\mu \circ L^{-1}\big) \quad \forall \mu \in M\big([0, 1]\big).$$

For every $c \in C$, let $\widetilde{F}_c := LF_cL^{-1}$. For $k \in \{0, 1, 2\}$, we define $\widetilde{G}_k(x_1, \dots, x_m) := LG_k(L^{-1}(x_1), \dots, L^{-1}(x_m))$. Define $\widetilde{\mathcal{W}}_0 := [0, 1]^m$ and inductively for $k \in \{1, 2\}$,

$$\widetilde{\mathcal{W}}_k := \big\{\big(\widetilde{F}_c(x_1), \dots, \widetilde{F}_c(x_m)\big) : c \in C, (x_1, \dots, x_m) \in \widetilde{\mathcal{W}}_{k-1}\big\}. \tag{39}$$

By definition, $\widetilde{G}_k$, $k \in \{0, 1, 2\}$, has domain $\widetilde{\mathcal{W}}_k$. Also, $\widetilde{\mathcal{W}}_0 \supset \widetilde{\mathcal{W}}_1 \supset \widetilde{\mathcal{W}}_2$. As in Section B.1, we use the Abel function to prove that $\widetilde{G}_0$ is translation-invariant.

LEMMA 6—Translation Invariance $\widetilde{G}_0$: *For every* $(x_1, \dots, x_m) \in \widetilde{\mathcal{W}}_0, \alpha \in (-1, 1)$ *such that* $(\alpha + x_1, \dots, \alpha + x_m) \in \widetilde{\mathcal{W}}_0$, *we have* $\widetilde{G}_0(\alpha + x_1, \dots, \alpha + x_m) = \alpha + \widetilde{G}_0(x_1, \dots, x_m)$.

PROOF: Let $(x_1, \dots, x_m)$ and α be as in the statement of the lemma. Let $y_i = L^{-1}(x_i)$ for $i = 1, \dots, m$. Then,

$$\begin{aligned}\widetilde{G}_0(\alpha + x_1, \dots, \alpha + x_m) &= LG_0\big(L^{-1}\big(\alpha + L(y_1)\big), \dots, L^{-1}\big(\alpha + L(y_m)\big)\big) \\ &= LG_0\big(f^\alpha(y_1), \dots, f^\alpha(y_m)\big) = Lf^\alpha\big(G_0(y_1, \dots, y_m)\big) \\ &= L\big(G_0(y_1, \dots, y_m)\big) + \alpha = \widetilde{G}_0(x_1, \dots, x_m) + \alpha. \qquad \textit{Q.E.D.}\end{aligned}$$

Because of the translation invariance of G_0, we can define a function ϕ_0 by

$$\phi_0(x_2 - x_1) := \widetilde{G}_0(x_1, x_2, x_2, \dots, x_2) - x_1, \quad \forall x_1, x_2 \in [0, 1]. \tag{40}$$

In Lemma S.5, we prove that ϕ_0 is continuous and strictly increasing and that $j - \phi_0$ is strictly decreasing.

[37]Section S.1.9 in the Supplemental Material shows how to modify the proof if either $f^1(0) = 0$ or $f^{-1}(1) = 1$.

B.6. *Two Linear Distributivity Equations*

We show in Section S.1.5 of the Supplemental Material that a result similar to Lemma 6, which proved the translation invariance of $\widetilde{G}_0$, holds for the functions $\widetilde{G}_1$ and $\widetilde{G}_2$ defined in Section B.5. The only difference is due to the fact that $\widetilde{G}_1$ and $\widetilde{G}_2$ are only defined on the non-rectangular domains $\bigcup_{c\in C}\widetilde{A}_c^m$ and $\bigcup_{c\in C}\widetilde{A}_{cc}^m$, respectively, where, for all $c \in C$,

$$\widetilde{A}_c^m := \{(\widetilde{F}_c(x_1), \dots, \widetilde{F}_c(x_m) : (x_1, \dots, x_m) \in [0, 1]^m\},$$
$$\widetilde{A}_{cc}^m := \{(\widetilde{F}_c(x_1), \dots, \widetilde{F}_c(x_m) : (x_1, \dots, x_m) \in \widetilde{A}_c^m\}.$$

For every $c \in C$, the functions $\widetilde{G}_1$ and $\widetilde{G}_2$ are translation-invariant on $\widetilde{A}_c^m$ and $\widetilde{A}_{cc}^m$, respectively. This makes it possible to define functions ϕ_1^c and ϕ_2^c by $\phi_1^c(x_2 - x_1) := \widetilde{G}_1(x_1, x_2, \dots, x_2) - x_1$ for all x_1, x_2 such that $(x_1, x_2, \dots, x_2) \in \widetilde{A}_c^m$, and $\phi_2^c(x_2 - x_1) := \widetilde{G}_2(x_1, x_2, \dots, x_2) - x_1$ for all x_1, x_2 such that $(x_1, x_2, \dots, x_2) \in \widetilde{A}_{cc}^m$. The functions $\widetilde{G}_0, \widetilde{G}_1, \widetilde{G}_2$, and $\widetilde{F}_c$ satisfy analogues of the equations in (35). Combining these equations with the definitions of $\widetilde{F}_c$, ϕ_0, ϕ_1^c, and ϕ_2^c, we obtain

$$\widetilde{F}_c\big(x_1 + \phi_0(x_2 - x_1)\big) = \widetilde{F}_c(x_1) + \phi_1^c\big(\widetilde{F}_c(x_2) - \widetilde{F}_c(x_1)\big), \tag{41}$$

$$\widetilde{F}_c\big(x_1 + \phi_1^c(x_2 - x_1)\big) = \widetilde{F}_c(x_1) + \phi_2^c\big(\widetilde{F}_c(x_2) - \widetilde{F}_c(x_1)\big), \tag{42}$$

where the first equation holds for all $c \in C$ and $x_1, x_2 \in [0, 1]$, while the second holds for all c and x_1, x_2 such that $(x_1, x_2, \dots, x_2) \in \widetilde{A}_c^m$. Equations such as (41) and (42) were studied in Lundberg (1985). His results, Theorem 11.1 in particular, are applicable since all functions are continuous, $\widetilde{F}_c$, ϕ_0, ϕ_1^c, ϕ_2^c are strictly increasing, and $j - \phi_0$, $j - \phi_1^c$, $j - \phi_2^c$ are strictly decreasing (see Lemma S.5 and Section S.1.5). For any given $c \in C$, Theorem 11.1 in Lundberg (1985) shows that there are four cases for the functions $\widetilde{F}_c$, ϕ_1^c that solve (41). As in Lundberg (1985), we enumerate those cases: (a), (b), (c), (d). In addition, we let $\Omega_{a)}$ be the set of all $c \in C$ such that the functions $\widetilde{F}_c$, ϕ_1^c belong to case (a). The sets $\Omega_{b)}, \Omega_{c)}, \Omega_{d)}$ are defined analogously. Lemma S.8 uses continuity arguments to show that all but one of those sets are empty, meaning that the system in (41)–(42) is solved by functions that belong to the same set.

Cases (b) and (c) can be ruled out. Indeed, for some $c \in C$, equations (41) and (42) are linked by the functions $\widetilde{F}_c$ and ϕ_1^c, which appear in both equations but in a "different position." However, it is known from Lundberg (1985) that functions that appear in "different positions" have different functional forms, which rules out (b) and (c).

We are thus left with cases (a) and (b) which we refer to as the affine and non-affine case and which we study in detail in Section S.1.7. To summarize, the affine case corresponds to $\widetilde{F}_c(x) = u(c) + b(c)x$ for every $c \in C$ and $x \in [0, 1]$. Moreover, the functions $u, b : C \to \mathbb{R}$ are continuous and $b(C) \subset (0, 1)$. While $\widetilde{G}_0$ is translation-invariant (Lemma 6), it can be further shown to be scale-invariant when the function b is not constant. The non-affine case means that $\widetilde{F}_c(x) = \frac{1}{a}\log(u(c) + b(c)e^{ax})$, with $b(\cdot)$ not constant. Even though this formulation looks quite different, defining $\hat{F}_c := H\widetilde{F}H^{-1}$ and $\hat{G}_0 = H\widetilde{G}_0H^{-1}$ with $H(x) := e^{ax}$ leads again to the affine case, with a function $\hat{G}_0$ that is both scale- and translation-invariant.

B.7. *Concluding the Proof of Proposition 1*

The preceding arguments show that is always possible to renormalize the utility representation, so as to obtain an affine time aggregator, $W(c,x) = u(c) + b(c)x$, and a renormalized certainty equivalent ($\widetilde{G}_0$ in the affine case and $\hat{G}_0$ in the non-affine case) which is translation-invariant (Lemmas 6 and S.13), and furthermore scale-invariant when the function b is not constant (Lemmas S.10 and S.11). Recall from (33) that G_0 was defined by fixing $m > 1$ and a probability vector $(\pi_1, \dots, \pi_m)$ and projecting I onto $[0,1]^m$. Since m and $(\pi_1, \dots, \pi_m)$ were arbitrary, we obtain that the recursive representation (U, W, I) of $\succeq$ can be renormalized so that:

– case 1: $W(c,x) = u(c) + \beta x$ and I is translation-invariant on $M^f(\mathcal{U})$,

– case 2: $W(c,x) = u(c) + b(c)x$ and I is translation- and scale-invariant on $M^f(\mathcal{U})$,

where $\mathcal{U} := U(D)$ and $M^f(\mathcal{U})$ is the set of simple lotteries with prizes drawn from the interval $\mathcal{U}$. In the first case, $u : C \to \mathbb{R}$ is continuous and $\beta \in (0,1)$. In the second, $u, b : C \to \mathbb{R}$ are continuous and $b(C) \subset (0,1)$. Since $M^f(\mathcal{U})$ is dense in $M(\mathcal{U})$ and the certainty equivalent $I : M(\mathcal{U}) \to \mathcal{U}$ is continuous, we know that if I is translation-invariant on $M^f(\mathcal{U})$, then I is also translation-invariant on $M(\mathcal{U})$. An identical argument holds for scale invariance.

To conclude the proof, it remains to take full account of the implications of Axiom 6 (Deterministic Monotonicity). First, note that the main features of the representations (U, W, I) we have derived so far—that $W(c,x)$ is affine in x and that I is translation- and, in the appropriate case, also scale-invariant—are preserved under positive affine transformations of utility. It is therefore w.l.o.g. to assume that the representations are chosen so that $U(D) = [0,1]$. This normalization, which we maintain in the statement of Proposition 1, makes it possible to express the implications of Axiom 6 in terms of the representation. When discounting is exogenous, everything is standard in that Axiom 6 is equivalent to the strict monotonicity of the function $u : C \to \mathbb{R}$. When discounting is endogenous, Axiom 6 is equivalent to the strict monotonicity of the functions $u, u + b : C \to \mathbb{R}$, provided that $U(D) = [0,1]$. To see this, note that $U(D) = [0,1]$ is equivalent to $U(\underline{c}, \underline{c}, \dots) = 0$ and $U(\overline{c}, \overline{c}, \dots) = 1$. The latter imply that for all $c \in C$, we have $U(c, \underline{c}, \underline{c}, \dots) = u(c)$ and $U(c, \overline{c}, \overline{c}, \dots) = u(c) + b(c)$, from where the strict monotonicity of $u, u + b$ follows.

APPENDIX C: PROOF OF PROPOSITION 3

The first point of the proposition was proved in Chew and Epstein (1991). For the second point, when preferences are represented as in the first case of Proposition 1, we have

$$U\left(c_0, \bigoplus_{i=1}^{n} \pi_i (c_1, c_i, m_i)\right) = u(c_0) + \beta u(c_1) + \beta I\left(\beta \bigoplus_{i=1}^{n} \pi_i U(c_i, m_i)\right),$$

while

$$U\left(c_0, c_1, \bigoplus_{i=1}^{n} \pi_i (c_i, m_i)\right) = u(c_0) + \beta u(c_1) + \beta^2 I\left(\bigoplus_{i=1}^{n} \pi_i U(c_i, m_i)\right).$$

Preference for early resolution of uncertainty is thus obtained if and only if $I(\beta \bigoplus_{i=1}^{n} \pi_i U(c_i, m_i)) \geq \beta I(\bigoplus_{i=1}^{n} \pi_i U(c_i, m_i))$, for all $n > 0$, $(c_i, m_i) \in D$, and $(\pi_i) \in [0,1]^n$ such that $\sum_{i=1}^{n} \pi_i = 1$. By continuity of I, that is equivalent to $I(\beta\mu) \geq \beta I(\mu)$ for all $\mu \in M([0,1])$.

For preferences that can be represented as in the second case of Proposition 1, we immediately get $U(c_0, \bigoplus_{i=1}^n \pi_i(c_1, c_i, m_i)) = U(c_0, c_1, \bigoplus_{i=1}^n \pi_i(c_i, m_i))$, implying both preference for early and for late resolution of uncertainty.

For the last point, we know from Proposition 2 that the Kreps–Porteus case corresponds to having a certainty equivalent of the expected utility form, $I = \phi^{-1}E\phi$, with $\phi(x) = \frac{1-\exp(-kx)}{k}$ for some $k \in \mathbb{R}$, the linear case being obtained by continuity when $k = 0$. Since $\beta < 1$, $I(\beta\mu) \geq \beta I(\mu)$ (resp: $I(\beta\mu) \leq \beta I(\mu)$) is equivalent to $k \geq 0$ (resp. $k \leq 0$). From Kreps and Porteus (1978), this is known to imply a preference for early (resp. late) resolution of uncertainty.

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